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Université de Tlemcen, BP 119, 13000-Tlemcen, Algérie.***Abstract**

We investigate polynomial sets  $\{P_n\}_{n \geq 0}$  with generating power series of the form  $F(xt - R(t))$  and satisfying, for  $n \geq 0$ , the  $(d+1)$ -order recursion  $xP_n(x) = P_{n+1}(x) + \sum_{l=0}^d \gamma_n^l P_{n-l}(x)$ , where  $\{\gamma_n^l\}$  is a complex sequence for  $0 \leq l \leq d$ ,  $P_0(x) = 1$  and  $P_n(x) = 0$  for all negative integer  $n$ . We show that the formal power series  $R(t)$  is a polynomial of degree at most  $d+1$  if certain coefficients of  $R(t)$  are null or if  $F(t)$  is a generalized hypergeometric series. Moreover, for the  $d$ -symmetric case we demonstrate that  $R(t)$  is the monomial of degree  $d+1$  and  $F(t)$  is expressed by hypergeometric series.

**Keywords:** Generating functions,  $d$ -orthogonal polynomials; recurrence relations; generalized hypergeometric series.

**AMS Subject Classification:** 12E10, 33C47; 33C20

**1 Introduction**

In [1, 3, 4] the authors used different methods to show that the orthogonal polynomials defined by a generating function of the form  $F(xt - \alpha t^2)$  are the ultraspherical and Hermite polynomials. On the other hand, the author in [2] found (even if  $F$  is a formal power series) that the orthogonal polynomials are the ultraspherical, Hermite and Chebychev polynomials of the first kind. Motivated by the problem, posed in [2], of describing (all or just orthogonal) polynomials with generating functions  $F(xU(t) - R(t))$  we have generalized in [14] the above results by proving the following:

**Theorem 1** [14] *Let  $F(t) = \sum_{n \geq 0} \alpha_n t^n$  and  $R(t) = \sum_{n \geq 1} R_n t^n/n$  be formal power series where  $\{\alpha_n\}$  and  $\{R_n\}$  are complex sequences with  $\alpha_0 = 1$  and  $R_1 = 0$ . Define the polynomial set  $\{P_n\}_{n \geq 0}$  by*

$$F(xt - R(t)) = \sum_{n \geq 0} \alpha_n P_n(x) t^n. \quad (1)$$

*If this polynomial set (which is automatically monic) satisfies the three-term recursion relation*

$$\begin{cases} xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \omega_n P_{n-1}(x), & n \geq 0, \\ P_{-1}(x) = 0, \quad P_0(x) = 1 \end{cases} \quad (2)$$

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where  $\{\beta_n\}$  and  $\{\omega_n\}$  are complex sequences, then we have:

- a) If  $R_2 = 0$  and  $\alpha_n \neq 0$  for  $n \geq 1$ , then  $R(t) = 0$ ,  $F(t)$  is arbitrary and  $F(xt) = \sum_{n \geq 0} \alpha_n x^n t^n$  generates the monomials  $\{x^n\}_{n \geq 0}$ .
- b) If  $\alpha_1 R_2 \neq 0$ , then  $R(t) = R_2 t^2 / 2$  and the polynomial sets  $\{P_n\}_{n \geq 0}$  are the rescaled ultraspherical, Hermite and Chebychev polynomials of the first kind.

Note that, the polynomials in Theorem 1 which satisfy a three term recursion with complex coefficients are not necessary orthogonal with respect to a moment functional  $\mathcal{L}$ , i.e. for all non negative integers  $m, n$ ;  $\mathcal{L}(P_m(x)P_n(x)) = 0$  if  $m \neq n$  and  $\mathcal{L}(P_n^2(x)) \neq 0$ , see Definition 2.2 in [7].

**Remark 1**

- a) A polynomial set PS,  $\{P_n\}_{n \geq 0}$ , is such that  $\text{degree}(P_n) = n$ ,  $n \geq 0$ .
- b) A PS is called a monic PS if  $P_n(x) = x^n + \dots$ , for  $n \geq 0$ .
- c) The choice  $\alpha_0 = 1$  and  $R_1 = 0$  comes from the fact that the generating function  $\gamma_1 + \gamma_2 F((x + R_1)t - R(t)) = \gamma_1 + \gamma_2 \sum_{n \geq 0} \alpha_n P_n(x + R_1)t^n$ , with  $\gamma_1$  and  $\gamma_2$  constants, is also of type (1).

In the present paper, we are interested in monic PSs generated by (1) (with  $F(t)$  and  $R(t)$  as in Theorem 1) and satisfying higher order recurrence relations (3). For this purpose, we adopt the following definitions:

**Definition 1** Let  $d \in \mathbb{N}$ . A PS  $\{Q_n\}_{n \geq 0}$  is called a  $d$ -polynomial set  $d$ -PS if its corresponding monic PS  $\{P_n\}_{n \geq 0}$ , defined by  $P_n(x) = (\lim_{x \rightarrow +\infty} x^{-n} Q_n(x))^{-1} Q_n(x)$ ,  $n \geq 0$ , satisfies the  $(d + 1)$ -order recurrence relation:

$$\begin{cases} xP_n(x) = P_{n+1}(x) + \sum_{l=0}^d \gamma_n^l P_{n-l}(x), & n \geq 0, \\ P_0(x) = 1, \quad P_{-l}(x) = 0, & 1 \leq l \leq d \end{cases} \quad (3)$$

where

$$\{\gamma_n^l\}_{n \geq 0}, 0 \leq l \leq d, \text{ are complex sequences} \quad (4)$$

and

$$\{\gamma_n^d\}_{n \geq d} \text{ is not the null sequence, for } d \geq 1. \quad (5)$$

**Definition 2** Let  $\{P_n\}_{n \geq 0}$  be a  $d$ -PS. If the PS of the derivatives  $\{(n + 1)^{-1} P'_{n+1}\}_{n \geq 0}$  is also a  $d$ -PS, then  $\{P_n\}_{n \geq 0}$  is called a classical  $d$ -PS.

**Definition 3** [9] Let  $\omega = \exp(2i\pi/(d + 1))$ , where  $i^2 = -1$ . The PS  $\{P_n\}_{n \geq 0}$  is called  $d$ -symmetric if it fulfils:

$$P_n(\omega x) = \omega^n P_n(x), \quad n \geq 0. \quad (6)$$

**Remark 2** In Definition 1:

- a) For  $d \geq 1$ , the first terms  $\{\gamma_n^l\}_{0 \leq n < l \leq d}$  of the sequences  $\{\gamma_n^l\}_{n \geq 0}$ ,  $1 \leq l \leq d$ , are arbitrary.
- b) For  $d = 0$ , (3) becomes  $xP_n(x) = P_{n+1}(x) + \gamma_n^0 P_n(x)$ ,  $n \geq 0$ , with  $P_0(x) = 1$ . Here  $\{\gamma_n^0\}_{n \geq 0}$  can be the null sequence, so the set of monomials is a 0-PS.

An interesting class of  $d$ -PSs characterized by (3), with the additional condition  $\gamma_n^d \neq 0$  for  $n \geq d$ , are the  $d$ -orthogonal polynomial sets  $d$ -OPSs [12, 15]. In this context, the authors in [4] generalized the result stated in [1, 3] by showing the following:

**Theorem 2** [4] *The only  $d$ -OPSs generated by  $G((d+1)xt - t^{d+1})$  are the classical  $d$ -symmetric polynomials.*

Another contribution concerns  $d$ -OPSs with generating functions of Sheffer type, i.e. of the form  $A(t) \exp(xH(t))$ . We have

**Theorem 3** [16] *Let  $\rho_d(t) = \sum_{k=0}^d \tilde{\rho}_k t^k$  be a polynomial of degree  $d$  ( $\tilde{\rho}_d \neq 0$ ) and  $\sigma_{d+1}(t) = \sum_{k=0}^{d+1} \tilde{\sigma}_k t^k$  be a polynomial of degree less than or equal to  $d+1$ . The only PSs, which are  $d$ -orthogonal and also Sheffer PS, are generated by*

$$\exp\left(\int_0^t \frac{\rho_d(s)}{\sigma_{d+1}(s)} ds\right) \exp\left(x \int_0^t \frac{1}{\sigma_{d+1}(s)} ds\right) = \sum_{k=0}^{\infty} P_n(x) \frac{t^n}{n!} \quad (7)$$

with the conditions

$$\tilde{\sigma}_0(n\tilde{\sigma}_0 - \tilde{\rho}_d) \neq 0, \quad n \geq 1. \quad (8)$$

Note that Theorem 3 characterizes also the  $d$ -OPSs with generating functions of the form  $F(xH(t) - R(t))$  with  $F(t) = \exp(t)$ , since  $A(t) \exp(xH(t)) = F(xH(t) - R(t))$  where  $R(t) = -\ln(A(t))$ . For  $H(t) = t$  we meet the Appell case with the Hermite  $d$ -OPSs [8] generated by  $\exp(xt - \bar{\rho}_{d+1}(t))$  where  $\bar{\rho}_{d+1}(t)$  is a polynomial of degree  $d+1$ .

As a consequence of the results obtained in this paper, we give some generalizations of Theorems 1, 2 and the Appell case in Theorem 3 (see also [8]) to  $d$ -PS generated by (1).

After this short introduction, we give in section 2 some results for  $d$ -PSs generated by (1). Then in section 3 we show that the only  $d$ -symmetric  $d$ -PSs generated by (1) are the classical  $d$ -symmetric polynomials. For this later case, we give in section 4 the  $(d+1)$ -order recurrence relation (3) and the expression of  $F(t)$  by means of hypergeometric functions.

## 2 Some general results

The results in this section concern all  $d$ -PSs generated by (1). The central result is Proposition 2 below from which the other results arise. First we have

**Proposition 1** [14]

*Let  $\{P_n\}_{n \geq 0}$  be a PS generated by (1). Then we have*

$$\alpha_n x P'_n(x) - \sum_{k=1}^{n-1} R_{k+1} \alpha_{n-k} P'_{n-k}(x) = n \alpha_n P_n(x), \quad n \geq 1. \quad (9)$$

Secondly

**Proposition 2** Let  $\{P_n\}$  be a  $d$ -PS generated by (1) and satisfying (3), (4) and (5), with  $\alpha_n \neq 0$  for  $n \geq 1$ . Putting

$$a_n = \frac{\alpha_n}{\alpha_{n+1}}, \quad (n \geq 0) \text{ and } c_n^l = \frac{\alpha_n}{\alpha_{n-l}} \gamma_n^l, \quad (1 \leq l \leq d, \quad n \geq l),$$

then we have:

a)

$$\gamma_n^0 = 0, \quad \text{for } n \geq 0. \quad (10)$$

b)

$$\gamma_n^1 = \frac{R_2}{2} (na_n - (n-1)a_{n-1}), \quad \text{for } n \geq 1. \quad (11)$$

or equivalently

$$c_n^1 = \frac{R_2}{2} \left( n \frac{a_n}{a_{n-1}} - (n-1) \right), \quad \text{for } n \geq 1. \quad (12)$$

c)

$$c_n^2 = \frac{R_3}{3} \left( (n-1) \frac{a_n}{a_{n-2}} - (n-2) \right), \quad \text{for } n \geq 2. \quad (13)$$

d.i)

$$\begin{aligned} \frac{k+1}{n-k+1} a_{n-k} c_n^k &= R_{k+1} \left( a_n - \frac{n-k}{n-k+1} a_{n-k} \right) + \sum_{l=1}^{k-2} R_{k-l} \left( \frac{n+2}{n-l+1} c_n^l - \frac{n-k+l+1}{n-k+l+2} c_{n-k+l+1}^l \right) \\ &\quad - \sum_{l=1}^{k-2} \frac{l+1}{n-l+1} c_n^l c_{n-l}^{k-l-1} - \sum_{l=1}^{k-2} \frac{R_{l+1} R_{k-l}}{n-l+1}, \quad 3 \leq k \leq d, \quad n \geq k. \end{aligned} \quad (14)$$

d.ii)

$$\begin{aligned} R_{k+1} \left( a_n - \frac{n-k}{n-k+1} a_{n-k} \right) + \sum_{l=1}^d R_{k-l} \left( \frac{n+2}{n-l+1} c_n^l - \frac{n-k+l+1}{n-k+l+2} c_{n-k+l+1}^l \right) \\ - \sum_{l=k-1-d}^d \frac{l+1}{n-l+1} c_n^l c_{n-l}^{k-l-1} = \sum_{l=1}^{k-2} \frac{R_{l+1} R_{k-l}}{n-l+1}, \quad d+1 \leq k \leq 2d+1, \quad n \geq k. \end{aligned} \quad (15)$$

d.iii)

$$\begin{aligned} R_{k+1} \left( a_n - \frac{n-k}{n-k+1} a_{n-k} \right) + \sum_{l=1}^d R_{k-l} \left( \frac{n+2}{n-l+1} c_n^l - \frac{n-k+l+1}{n-k+l+2} c_{n-k+l+1}^l \right) \\ = \sum_{l=1}^{k-2} \frac{R_{l+1} R_{k-l}}{n-l+1}, \quad k \geq 2d+2, \quad n \geq k. \end{aligned} \quad (16)$$

**Proof:** By differentiating (3) we get

$$xP'_n(x) + P_n(x) = P'_{n+1}(x) + \sum_{l=0}^d \gamma_n^l P'_{n-l}(x). \quad (17)$$

Then by making the operations  $n\alpha_n Eq(17) + Eq(9)$  and  $Eq(9) - \alpha_n Eq(17)$  we obtain, respectively,

$$(n+1)\alpha_n xP'_n(x) = n\alpha_n \left( P'_{n+1}(x) + \sum_{l=0}^d \gamma_n^l P'_{n-l}(x) \right) + \sum_{k=1}^{n-1} R_{k+1} \alpha_{n-k} P'_{n-k}(x) \quad (18)$$

and

$$(n+1)\alpha_n P_n(x) = \alpha_n \left( P'_{n+1}(x) + \sum_{l=0}^d \gamma_n^l P'_{n-l}(x) \right) - \sum_{k=1}^{n-1} R_{k+1} \alpha_{n-k} P'_{n-k}(x). \quad (19)$$

Inserting (9) in the left-hand side of the equation  $Eq(19)$  multiplied by  $x$  we obtain

$$(n+1)\alpha_n \left( P_{n+1} + \sum_{l=0}^d \gamma_n^l P_{n-l}(x) \right) = \alpha_n \left( xP'_{n+1}(x) + \sum_{l=0}^d \gamma_n^l xP'_{n-l}(x) \right) - \sum_{k=1}^{n-1} R_{k+1} \alpha_{n-k} xP'_{n-k}(x). \quad (20)$$

Using (19) and (18) respectively in the left hand side and right hand side of (20) we get

$$\begin{aligned} & \frac{n+1}{n+2} \alpha_n \left( P'_{n+2}(x) + \sum_{l=0}^d \gamma_{n+1}^l P'_{n+1-l}(x) \right) - \frac{n+1}{n+2} \frac{\alpha_n}{\alpha_{n+1}} \sum_{k=1}^n R_{k+1} \alpha_{n+1-k} P'_{n+1-k}(x) \\ & + (n+1)\alpha_n \sum_{l=0}^d \gamma_n^l \frac{1}{n+1-l} P'_{n+1-l}(x) + (n+1)\alpha_n \sum_{l=0}^d \gamma_n^l \frac{1}{n+1-l} \sum_{l'=0}^d \gamma_{n-l}^{l'} P'_{n-l-l'}(x) \\ & - (n+1)\alpha_n \sum_{l=0}^d \gamma_n^l \frac{1}{n+1-l} \frac{1}{\alpha_{n-l}} \sum_{k=1}^{n-l-1} R_{k+1} \alpha_{n-l-k} P'_{n-k-l}(x) = \frac{n+1}{n+2} \alpha_n \left( P'_{n+2}(x) + \sum_{l=0}^d \gamma_{n+1}^l P'_{n+1-l}(x) \right) \\ & + \frac{1}{n+2} \frac{\alpha_n}{\alpha_{n+1}} \sum_{k=1}^n R_{k+1} \alpha_{n+1-k} P'_{n+1-k}(x) + \alpha_n \sum_{l=0}^d \gamma_n^l \frac{n-l}{n+1-l} P'_{n+1-l}(x) \\ & + \alpha_n \sum_{l=0}^d \gamma_n^l \frac{n-l}{n+1-l} \sum_{l'=0}^d \gamma_{n-l}^{l'} P'_{n-l-l'}(x) + \alpha_n \sum_{l=0}^d \gamma_n^l \frac{1}{n+1-l} \frac{1}{\alpha_{n-l}} \sum_{k=1}^{n-l-1} R_{k+1} \alpha_{n-l-k} P'_{n-k-l}(x) \\ & - \sum_{k=1}^{n-1} R_{k+1} \frac{n-k}{n+1-k} \alpha_{n-k} P'_{n+1-k}(x) - \sum_{k=1}^{n-1} R_{k+1} \frac{n-k}{n+1-k} \alpha_{n-k} \sum_{l=0}^d \gamma_{n-k}^l P'_{n-k-l}(x) \\ & - \sum_{k=1}^{n-1} R_{k+1} \frac{1}{n+1-k} \sum_{k'=1}^{n-k-1} R_{k'+1} \alpha_{n-k-k'} P'_{n-k-k'}(x). \end{aligned} \quad (21)$$

It follows that

$$\begin{aligned}
& -\frac{\alpha_n}{\alpha_{n+1}} \sum_{k=1}^n R_{k+1} \alpha_{n+1-k} P'_{n+1-k}(x) + \alpha_n \sum_{l=0}^d \gamma_n^l \frac{l+1}{n+1-l} P'_{n+1-l}(x) + \alpha_n \sum_{l=0}^d \gamma_n^l \frac{l+1}{n+1-l} \sum_{l'=0}^d \gamma_{n-l}^{l'} P'_{n-l-l'}(x) \\
& -(n+2) \alpha_n \sum_{l=0}^d \gamma_n^l \frac{1}{n+1-l} \frac{1}{\alpha_{n-l}} \sum_{k=1}^{n-l-1} R_{k+1} \alpha_{n-l-k} P'_{n-k-l}(x) = - \sum_{k=1}^{n-1} R_{k+1} \frac{n-k}{n+1-k} \alpha_{n-k} P'_{n+1-k}(x) \\
& - \sum_{k=1}^{n-1} R_{k+1} \frac{n-k}{n+1-k} \alpha_{n-k} \sum_{l=0}^d \gamma_{n-k}^l P'_{n-k-l}(x) - \sum_{k=1}^{n-1} R_{k+1} \frac{1}{n+1-k} \sum_{k'=1}^{n-k-1} R_{k'+1} \alpha_{n-k-k'} P'_{n-k-k'}(x) \quad (22)
\end{aligned}$$

a) By comparing the coefficients of  $P'_{n+1}(x)$  in the both sides of (22) we obtain

$$\frac{1}{n+1} \alpha_n \gamma_n^0 = 0, \quad \text{for } n \geq 0,$$

and then

$$\gamma_n^0 = 0, \quad \text{for } n \geq 0.$$

b) Equating the coefficients of  $P'_n(x)$  in the both sides of the equation (22) gives

$$\frac{2}{n} \alpha_n \gamma_n^1 = R_2 \frac{\alpha_n}{\alpha_{n+1}} \alpha_n - R_2 \frac{n-1}{n} \alpha_{n-1}, \quad \text{for } n \geq 1,$$

which can be written as

$$\gamma_n^1 = n a_n - (n-1) a_{n-1}, \quad \text{for } n \geq 1.$$

Now by equating the coefficients of  $P'_{n+1-k}(x)$  for  $k \geq 2$  in the both sides of the equation (22) we obtain

$$\begin{aligned}
& \frac{k+1}{n+1-k} \alpha_n \gamma_n^k + R_{k+1} \alpha_{n-k} \frac{n-k}{n+1-k} - R_{k+1} \frac{\alpha_n}{\alpha_{n+1}} \alpha_{n+1-k} + \alpha_n \sum_{l=1}^d \gamma_n^l \frac{l+1}{n+1-l} \sum_{l'=1}^d \gamma_{n-l}^{l'} \delta_{l+l'}^{k-1} \\
& -(n+2) \alpha_n \sum_{l=1}^d \gamma_n^l \frac{1}{n+1-l} \sum_{k'=1}^{n-l-1} R_{k'+1} \frac{\alpha_{n-l-k'}}{\alpha_{n-l}} \delta_{l+k'}^{k-1} + \sum_{k'=1}^n R_{k'+1} \alpha_{n-k'} \frac{n-k'}{n+1-k'} \sum_{l=1}^d \gamma_{n-k'}^l \delta_{k'+l}^{k-1} \\
& + \sum_{k''=1}^n R_{k''+1} \frac{1}{n+1-k''} \sum_{k'=1}^{n-k''} R_{k'+1} \alpha_{n-k'-k''} \delta_{k'+k''}^{k-1} = 0, \quad \text{for } n \geq k. \quad (23)
\end{aligned}$$

then by taking  $k = 2$  in (23) we retrieve c) and by considering  $3 \leq k \leq d$ ,  $d+1 \leq k \leq 2d+1$  and  $k \geq 2d+2$  we obtain d.i.), d.ii.) and d.iii.) respectively. ■

In the following corollaries we adopt the same conditions and notations as in Proposition 2.

**Corollary 1** *If  $R_2 = R_3 = \dots = R_{d+1} = 0$  and  $\alpha_n \neq 0$  for  $n \geq 1$ , then  $R(t) = 0$ ,  $F(t)$  is arbitrary and  $F(xt) = \sum_{n \geq 0} \alpha_n x^n t^n$  generates the monomials  $\{x^n\}_{n \geq 0}$ .*

**Proof:** As  $R_1 = R_2 = \dots = R_{d+1} = 0$ , it is enough to show by induction that  $R_n = 0$  for  $n \geq d+2$ . For  $n = 1, 2, \dots, d+2$ , the equation (9) gives  $P_1(x) = x$ ,  $P_2(x) = x^2, \dots, P_{d+1}(x) = x^{d+1}$  and  $P_{d+2}(0) = \frac{-R_{d+2}\alpha_1}{(d+2)\alpha_{d+2}}$ . But according to equation (3), for  $n = d+1$ ,  $P_{d+2}(0) = 0$  and then  $R_{d+2} = 0$ .

Now assume that  $R_k = 0$  for  $d+2 \leq k \leq n-1$ . According to (9) we have, for  $d+2 \leq k \leq n-1$ ,  $P_k(0) = 0$  and  $P_n(0) = -\frac{R_n\alpha_1}{n\alpha_n}$ . On other hand, by the shift  $n \rightarrow n-1$  in (3) we have  $P_n(0) = 0$  and thus  $R_n = 0$ . As  $R(t) = 0$ , the generating function (1) reduces to  $F(xt) = \sum_{n \geq 0} \alpha_n x^n t^n$  which generates the monomials with  $F(t)$  arbitrary. ■

**Corollary 2** *If  $R_{d+2} = R_{d+3} = \dots = R_{2d+2} = 0$  then  $R(t) = R_2 t^2/2 + R_3 t^3/3 + \dots + R_{d+1} t^{d+1}/(d+1)$ .*

**Proof:** We will use (16) and proceed by induction on  $k$  to show that  $R_k = 0$  for  $k \geq 2d+3$ . Indeed  $k = 2d+2$  and  $n = 2d+2$  in (16) leads to  $a_{2d+2} R_{2d+3} = 0$  and since  $a_n \neq 0$  we get  $R_{2d+3} = 0$ . Suppose that  $R_{2d+3} = R_{2d+4} = \dots = R_k = 0$ , then for  $n = k$  the equation (16) gives  $a_k R_{k+1} = 0$  and finally  $R_{k+1} = 0$ . ■

**Corollary 3** *If  $R_{\kappa+d+1} = \dots = R_{\kappa+1} = R_\kappa = R_{\kappa-1} = \dots = R_{\kappa-d} = 0$  for some  $\kappa \geq 3d+3$ , then  $R_{d+2} = R_{d+3} = \dots = R_{2d+2} = 0$ .*

**Proof:**

• Let  $k = \kappa$  in (16), then for  $n \geq \kappa$  the fraction  $\sum_{l=1}^{\kappa-2} \frac{R_{l+1} R_{\kappa-l}}{n-l+1}$ , as function of integer  $n$ , is null even for real  $n$ . So,

$$\lim_{x \rightarrow l-1} (x-l+1) \sum_{s=1}^{\kappa-2} \frac{R_{s+1} R_{\kappa-s}}{x-s+1} = R_{l+1} R_{\kappa-l} = 0, \text{ for } 1 \leq l \leq \kappa-2 \quad (24)$$

which is  $R_{d+2} R_{\kappa-d-1} = 0$  when  $l = d+1$ . Supposing  $R_{d+2} \neq 0$  leads to  $R_{\kappa-d-1} = 0$ . So  $R_{\kappa+d} = R_{\kappa+d-1} = \dots = R_{\kappa-d} = R_{\kappa-d-1} = 0$  and with the same procedure we find  $R_{\kappa-d-2} = 0$ . Going so on till we arrive at  $R_{d+2} = 0$  which contradicts  $R_{d+2} \neq 0$ .

• By taking successively  $k = \kappa + r, \kappa + r - 1, \dots, \kappa$  in (16), for  $1 \leq r \leq d$ , we find

$$\begin{aligned} R_{l+1} R_{\kappa+r-l} &= 0 \text{ for } 1 \leq l \leq \kappa + r - 2, \\ R_{l+1} R_{\kappa+r-1-l} &= 0 \text{ for } 1 \leq l \leq \kappa + r - 3, \\ &\vdots \\ R_{l+1} R_{\kappa-l} &= 0 \text{ for } 1 \leq l \leq \kappa - 2. \end{aligned}$$

If  $R_{d+2+r} \neq 0$  then by taking  $l = d+1+r$  we get  $R_{\kappa-d-1} = R_{\kappa-d-2} = \dots = R_{\kappa-d-r-1} = 0$ . So  $R_{\kappa+d-r} = R_{\kappa+d-r-1} = \dots = R_{\kappa-d-r-1} = 0$  and with the same procedure we find  $R_{\kappa-d-r-2} = R_{\kappa-d-r-3} = \dots = R_{\kappa-d-2r-2} = 0$ . Going so on till we arrive at  $R_{d+2+r} = 0$  which contradicts  $R_{d+2+r} \neq 0$ . ■

**Corollary 4** *If  $a_n$  is a rational function of  $n$  then  $R_{d+2} = R_{d+3} = \dots = R_{2d+2} = 0$ .*

**Proof:** From (12), (13) and (14) observe that  $c_n^l$  will also be a rational function of  $n$ . Then it follows that, in (16), two fractions are equal for natural numbers  $n \geq k$ ,  $k \geq 2d+2$ , and consequently will be for real numbers  $n$ . If we denote by  $N_s(G(x))$  the number of singularities of a rational function  $G(x)$  then we can easily verify, for all rational functions  $G$  and  $\tilde{G}$  of  $x$  and a constant  $a \neq 0$ , that:

- a)  $N_s(G(x+a)) = N_s(G(x))$ ,
- b)  $N_s(aG(x)) = N_s(G(x))$ ,
- c)  $N_s(G(x) + \tilde{G}(x)) \leq N_s(G(x)) + N_s(\tilde{G}(x))$ .

Using property a) of  $N_s$  we have

$$N_s \left( \frac{n-k}{n-k+1} a_{n-k} \right) = N_s \left( \frac{n}{n+1} a_n \right) \text{ and } N_s \left( \frac{n-k+l+1}{n-k+l+2} c_{n-k+l+1}^l \right) = N_s \left( \frac{n}{n+1} c_n^l \right).$$

According to properties b) and c) of  $N_s$ , the  $N_s$  of the left-hand side of (15) is finite and independent of  $k$ . Thus, the right-hand side of (15) has a finite number of singularities which is independent of  $k$ . As consequence there exists a  $k_1 \geq 3d+3$  for which  $R_{l+1}R_{k-l} = 0$  for all  $k \geq k_1 - d - 1$  and  $k_1 - d - 1 \leq l \leq k$ . According to Corollary 1, there exists a  $k_0$  such that  $2 \leq k_0 \leq d+1$  and  $R_{k_0} \neq 0$ . So, taking successively  $k = k_0 + l$  with  $l = k_1 + d, k_1 + d - 1, \dots, k_1 - d - 1$  we get  $R_{k_1+d+1} = R_{k_1+d} = \dots = R_{k_1-d} = 0$ . Then, by Corollary 3 we have  $R_{d+2} = R_{d+3} = \dots = R_{2d+2} = 0$ . ■

The fact that  $a_n$  is a rational function of  $n$  means that  $F(\epsilon z) = \sum_{n \geq 0} \alpha_n (\epsilon z)^n$  (where  $\epsilon$  is the quotient of the leading coefficients of the numerator and the denominator of  $a_n$ ) is a generalized hypergeometric series, i.e. of the form:

$${}_pF_q \left( \begin{matrix} (\mu_l)_{l=1}^p \\ (\nu_l)_{l=1}^q \end{matrix} ; z \right) = {}_pF_q \left( \begin{matrix} \mu_1, \mu_2, \dots, \mu_p \\ \nu_1, \nu_2, \dots, \nu_q \end{matrix} ; z \right) = \sum_{n \geq 0} \frac{(\mu_1)_n (\mu_2)_n \dots (\mu_p)_n}{(\nu_1)_n (\nu_2)_n \dots (\nu_q)_n} \frac{z^n}{n!} \quad (25)$$

where  $(\mu_l)_{l=k}^p$  denotes the array of complex parameters  $\mu_k, \mu_{k+1}, \dots, \mu_p$ , and if  $k > p$  we take the convention that  $(\mu_l)_{l=k}^p$  is the empty array. The symbol  $(\mu)_n$  stands for the shifted factorials, i.e.

$$(\mu)_0 = 1, \quad (\mu)_n = \mu(\mu+1) \dots (\mu+n-1), \quad n \geq 1. \quad (26)$$

As an interesting consequence, from Corollary 4 and Corollary 2 we state the following result, which can be interpreted as a generalization of the Appell case in the above Theorem 3 (see also [8]):

**Theorem 4** *Let  $\{P_n\}$  be a  $d$ -PS generated by (1) with  $F(z)$  a generalized hypergeometric series. Then  $R(t) = R_2 t^2/2 + R_3 t^3/3 + \dots + R_{d+1} t^{d+1}/(d+1)$ .*

**Proof:**  $F(z) = \sum_{n \geq 0} \alpha_n z^n$  has the form (25). Then  $a_n = \alpha_n / \alpha_{n+1}$  is a rational function of  $n$ , since  $(\mu)_{n+1}/(\mu)_n = n + \mu$ . The use of Corollary 4 and Corollary 2 completes the proof. ■

**Corollary 5** *Let  $R(t) = R_2 t^2/2 + R_3 t^3/3 + \dots + R_{d+1} t^{d+1}/(d+1)$ . Then,*

i) *If  $R_{d+1} = 0$  we have  $c_{n+d}^d c_n^d = 0$ , for  $n \geq d+1$ .*

ii) *If  $R_{d+1} \neq 0$  then*

$$c_n^d = \frac{R_{d+1}}{d+1} \left( (n+1) \frac{b_{n-d}}{b_n} - (n-d) \right), \quad \text{for } n \geq d+1, \quad (27)$$



where  $b_{md+r} = (b_{d+r} - b_r)m + b_r$ , for  $m \geq 0$ ,  $1 \leq r \leq d$ .

iii) The  $\{c_n^m\}_{1 \leq m \leq d-1}$  can be calculated recursively by solving the following  $d$ -order linear difference equations:

$$\begin{aligned} & \frac{1}{n-m+1} \left( R_{d+1}(n+2) - (n-d)c_{n-m}^d \right) c_n^m - \frac{1}{n-d+1} \left( R_{d+1}(n-d) + (d+1)c_n^d \right) c_{n-d}^m + \\ & + \sum_{l=m+1}^d R_{m+d+1-l} \left( \frac{n+2}{n-l+1} c_n^l - \frac{n-m-d+l}{n-m-d+l+1} c_{n-m-d+l}^l \right) - \sum_{l=m+1}^{d-1} \frac{l+1}{n-l+1} c_n^l c_{n-l}^{m+d-l} \\ & = \sum_{l=m}^{d-1} \frac{R_{l+1}R_{m+d+1-l}}{n-l+1}, \quad 1 \leq m \leq d-1, \quad n \geq m+d+1. \end{aligned} \quad (28)$$

**Proof:**

**The proof of i)**

Put  $k = 2d+1$  in (15) to get the following Riccati equation for  $\{c_n^d\}$ :

$$R_{d+1} \left( (n+2)c_n^d - (n-d)c_{n-d}^d \right) - (d+1)c_n^d c_{n-d}^d - R_{d+1}^2 = 0, \quad \text{for } n \geq 2d+1. \quad (29)$$

By taking  $R_{d+1} = 0$  in (29), i) follows immediately.

**The proof of ii)**

Substituting (27) in (29) we find the  $2d$ -linear homogeneous equation

$$b_n - 2b_{n-d} + b_{n-2d} = 0, \quad \text{for } n \geq 2d+1. \quad (30)$$

By writing  $n = md + r$ , where  $m, r$  are natural numbers with  $1 \leq r \leq d$ , the equation (30) can be solved by summing twice to find that

$$b_{md+r} = (b_{d+r} - b_r)m + b_r, \quad \text{for } m \geq 0, \quad 1 \leq r \leq d.$$

**The proof of iii)**

Since  $d+1 \leq k \leq 2d+1$  we have  $R_{k+1} = 0$  and  $R_{k-l} = 0$  for  $l \leq k-2-d$ . So, we can write (15) as

$$\begin{aligned} & \sum_{l=k-d}^d R_{k-l} \left( \frac{n+2}{n-l+1} c_n^l - \frac{n-k+l+1}{n-k+l+2} c_{n-k+l+1}^l \right) + R_{d+1} \left( \frac{n+2}{n-k+d+2} c_n^{k-1-d} - \frac{n-d}{n-d+1} c_{n-d}^{k-1-d} \right) \\ & - \sum_{l=k-d}^{d-1} \frac{l+1}{n-l+1} c_n^l c_{n-l}^{k-l-1} - \frac{k-d}{n-k+d+2} c_n^{k-1-d} c_{n-k+1+d}^d - \frac{d+1}{n-d+1} c_n^d c_{n-d}^{k-1-d} = \\ & = \sum_{l=k-1-d}^{k-2} \frac{R_{l+1}R_{k-l}}{n-l+1}, \quad d+1 \leq k \leq 2d+1, \quad n \geq k. \end{aligned} \quad (31)$$

Putting  $m = k-d-1 \neq 0$  in (31) and rearranging we obtain (28). ■

**Remark 3** In the case of  $d$ -OPSs, in Corollary 5 the polynomial  $R(t)$  is of degree  $d+1$ . Otherwise (i.e.  $R_{d+1} = 0$ ), we have a contradiction with the regularity conditions  $\gamma_n^d \neq 0$ , for  $n \geq d$ .

**Corollary 6** *The  $d$ -PS is classical if and only if  $R(t) = R_2 t^2/2 + R_3 t^3/3 + \cdots + R_{d+1} t^{d+1}/(d+1)$  with  $R_{d+1} \neq 0$ .*

**Proof:**

1) Assume that the  $d$ -PS is classical. From (18) and Definition 2 we have  $R_{k+1} = 0$  for  $d+1 \leq k \leq n-1$ . We get  $R(t)$  by taking  $n \geq 2d+2$  and using Corollary 2. Now we show that  $R_{d+1} \neq 0$ . Equation (18) becomes

$$xQ_n(x) = Q_{n+1}(x) + \sum_{l=1}^d \tilde{\gamma}_n^l Q_{n-l}(x), \quad n \geq 0, \quad (32)$$

where  $Q_n(x) = (n+1)^{-1} P'_{n+1}(x)$  and

$$\tilde{\gamma}_n^l = \frac{n+1-l}{n+2} \left( \gamma_{n+1}^l + \frac{R_{l+1} \alpha_{n+l-1}}{(n+1) \alpha_{n+1}} \right), \quad n \geq d. \quad (33)$$

From (27), if  $R_{d+1} = 0$  then  $\gamma_{n+1}^d = 0$ , and (33) gives  $\tilde{\gamma}_n^d = 0$ , for  $n \geq d$ . So,  $\{Q_n\}$  is not a  $d$ -PS which contradicts the fact that  $\{P_n\}$  is classical (see Definition 2).

2) Assume that  $R(t) = R_2 t^2/2 + R_3 t^3/3 + \cdots + R_{d+1} t^{d+1}/(d+1)$  with  $R_{d+1} \neq 0$ , then the PS of the derivatives  $\{Q_n\}$  satisfy (32) and are generated by  $F'(xt - R(t)) = \sum_{n \geq 0} (n+1) \alpha_{n+1} Q_n(x) t^n$ . Using Corollary 5 we find that  $\tilde{c}_n^d := (n+1) \alpha_{n+1} \tilde{\gamma}_n^d / ((n-d+1) \alpha_{n-d+1})$  satisfies (29). And according to the same expression (29), we should have, if  $c_n^d = 0$  or  $\gamma_n^d = 0$  (for  $n \geq d+1$ ),  $R_{d+1} = 0$ . Therefore, there exists for  $\tilde{c}_n^d$ , since  $R_{d+1} \neq 0$ , a  $n_0 \geq d+1$  such that  $\tilde{c}_{n_0}^d \neq 0$  or  $\tilde{\gamma}_{n_0}^d \neq 0$ . This means that  $\{P_n\}$  is classical. ■

### 3 The $d$ -symmetric case

The main result of this section is the following:

**Theorem 5** *If  $\{P_n\}$  is a  $d$ -symmetric  $d$ -PS generated by (1) then  $R(t) = R_{d+1} t^{d+1}/(d+1)$ .*

Theorem 5 generalizes Theorem 1 and Theorem 2 mentioned above. Its proof is quite similar to that of Theorem 1 in [14] and it requires the following Lemmas.

**Lemma 1** *If  $\{P_n\}$  is a  $d$ -symmetric  $d$ -PS generated by (1) then*

$$R(t) = \sum_{k \geq 1} \frac{R_{k(d+1)}}{k(d+1)} t^{k(d+1)}. \quad (34)$$

**Proof:** Let  $\{P_n\}$  be a  $d$ -symmetric  $d$ -PS satisfying (3) and generated by (1). Then it has, according to Definition 3, the property

$$P_n(\omega x) = \omega^n P_n(x), \quad (35)$$

where  $\omega = \exp(2\pi i/(d+1))$ . It follows that (3) becomes [9]

$$\begin{cases} xP_n(x) = P_{n+1}(x) + \gamma_n^d P_{n-d}(x), & n \geq 0, \\ P_{-n}(x) = 0, & 1 \leq n \leq d, \text{ and } P_0(x) = 1. \end{cases} \quad (36)$$

Let us show that  $R_k = 0$  when  $k$  is not a multiple of  $d + 1$ . First we replace  $x$  by  $\omega x$  in (9) and use (35) with  $P'_n(\omega x) = \omega^{n-1}P'_n(x)$  to get

$$\alpha_n x P'_n(x) - \sum_{k=1}^{n-1} R_{k+1} \alpha_{n-k} \omega^{-k-1} P'_{n-k}(x) = n \alpha_n P_n(x), \quad n \geq 2. \quad (37)$$

Subtracting (37) from (9) gives

$$\sum_{k=1}^{n-1} R_{k+1} \alpha_{n-k} (1 - \omega^{-k-1}) P'_{n-k}(x) = 0, \quad n \geq 2 \quad (38)$$

which leads to

$$R_k \alpha_{n-k+1} (1 - \omega^{-k}) = 0, \quad \text{for } 2 \leq k \leq n, \quad n \geq 2.$$

Since  $\omega^k \neq 1$ , provided  $k$  is not a multiple of  $d + 1$ , gives the result.  $\blacksquare$

By Lemma 1 and putting  $T_k = R_{k(d+1)}$  for  $k \geq 0$ , the equations in Proposition 2 simplify to particular forms. Indeed, from (12), (13) and (14) we get

$$c_n^d = \frac{T_1}{d+1} \left( (n-d+1) \frac{a_n}{a_{n-d}} - (n-d) \right), \quad \text{for } n \geq d. \quad (39)$$

The equation (15), with  $k = 2d + 1$ , becomes

$$\begin{aligned} & T_2 \left( a_n - \frac{n-2d-1}{n-2d} a_{n-2d-1} \right) + T_1 \left( \frac{n+2}{n-d+1} c_n^d - \frac{n-d}{n-d+1} c_{n-d}^d \right) \\ & - \frac{d+1}{n-d+1} c_n^d c_{n-d}^d = \frac{T_1^2}{n-d+1}, \quad n \geq 2d+1, \end{aligned} \quad (40)$$

which by (39) takes the form

$$\frac{(d+1)T_2}{T_1^2} \left( 1 - \frac{n-2d-1}{n-2d} \frac{a_{n-2d-1}}{a_n} \right) = \frac{n+1}{a_n} - \frac{2(n-d+1)}{a_{n-d}} + \frac{n-2d+1}{a_{n-2d}}, \quad \text{for } n \geq 2d+1. \quad (41)$$

Finally, the equation (16) simplifies to

$$\begin{aligned} & T_{k+1} \left( a_n - \frac{n-k(d+1)-d}{n-k(d+1)-d+1} a_{n-k(d+1)-d} \right) + T_k \left( \frac{n+2}{n-d+1} c_n^d - \frac{n-k(d+1)+1}{n-k(d+1)+2} c_{n-k(d+1)+1}^d \right) \\ & = \sum_{l=0}^{k-1} \frac{T_{l+1} T_{k-l}}{n-l(d+1)-d+1}, \quad k \geq 2, \quad n \geq k(d+1)+d. \end{aligned} \quad (42)$$

This equation will be denoted by  $E_k(n)$  in below.

**Lemma 2** *If  $T_2 = 0$  then  $R(t) = T_1 t^{d+1}/(d+1)$ .*

**Proof:** According to Corollary 2, if  $T_2 = R_{2(d+1)} = 0$  then  $R(t) = T_1 t^{d+1}/(d+1)$ , since in this case we have  $R_{d+2} = R_{d+3} = \dots = R_{2d+1} = 0$ .  $\blacksquare$

**Lemma 3** *If  $T_m = T_{m+1} = 0$  for some  $m \geq 3$ , then  $T_2 = 0$ .*

**Proof:**  $T_m = T_{m+1} = 0$  means that  $R_{(d+1)m} = R_{(d+1)(m+1)} = 0$ . Also by Lemma 1, we have  $R_{(d+1)m+d} = R_{(d+1)m+d-1} = \dots = R_{(d+1)m+1} = 0$  and  $R_{(d+1)m-1} = R_{(d+1)m-2} = \dots = R_{(d+1)m-d} = 0$  which represents the condition of corollary 3 with  $\kappa = (d+1)m \geq 3(d+1)$  and therefore gives  $R_{2d+2} = T_2 = 0$ . ■

**Lemma 4** *If  $T_\kappa = T_m = 0$  for some  $\kappa \neq m \geq 3$ , then  $T_2 = 0$ .*

**Proof:** The proof is similar to that of Corollary 7 in [14]. Let assume that  $T_{\kappa+1} \neq 0$  and  $T_{m+1} \neq 0$ , since if not, we apply Corollary 3. When  $m > \kappa$  and by using (42), the following operations

$$\begin{aligned} & [E_\kappa(n + (d+1)m + d)/T_{\kappa+1} - E_m(n + (d+1)m + d)/T_{m+1}] \\ & - [E_\kappa(n)/T_{\kappa+1} - E_m(n + (d+1)(m - \kappa) + d)/T_{m+1}] \end{aligned} \quad (43)$$

give

$$\left( \frac{n}{n+1} - \frac{n + (d+1)(m - \kappa)}{n + (d+1)(m - \kappa) + 1} \right) a_n = Q(n) \quad (44)$$

where  $Q(n)$  is a rational function of  $n$ . Consequently,  $a_n$  is a rational function of  $n$  and by Corollary 4 we have  $T_2 = 0$ . ■

**Lemma 5** *The following equality is true for  $k \geq 3$  and  $n \geq k(d+1) + 2d + 1$ .*

$$T_{k-1}D_{k+1}(a_n - \tilde{a}_{n-k(d+1)-2d-1}) - T_{k+1}D_k(a_{n-d-1} - \tilde{a}_{n-k(d+1)-d}) = \sum_{l=1}^{k-1} \frac{V_{k,l}}{n - l(d+1) - d + 1}, \quad (45)$$

where

- $D_{k,l} = T_k T_{k-l+1} - T_{k+1} T_{k-l}$ .
- $D_k = D_{k,1} = T_k^2 - T_{k+1} T_{k-1}$ .
- $V_{k,l} = \frac{T_1}{2} (T_l T_{k+1} D_{k-1,l-1} - T_{l+1} T_{k-1} D_{k,l})$ .
- $\tilde{a}_n = \frac{n}{n+1} a_n$ .

**Proof:** Just by making the following combinations it is easy to get (45):

$$T_{k+1} (T_{k-1} E_k(n) - T_k E_{k-1}(n - d - 1)) - T_{k-1} (T_k E_{k+1}(n) - T_{k+1} E_k(n - d - 1)).$$

To prove Theorem 5 it is sufficient, according to Lemma 2, to show that  $T_2 = 0$ . To this end, we will consider three cases:

**Case 1:** There exists  $k_0 \geq 3$  such that  $D_k \neq 0$  for  $k \geq k_0$ .

Considering Lemma 4, we can choose  $\tilde{k} \geq k_0$  such that  $T_k \neq 0$  for  $k \geq \tilde{k} - 1$ . Let define, for  $k \geq \tilde{k}$ ,  $\bar{D}_k = \frac{D_k}{T_{k-1}T_k}$  and  $\bar{E}_k(n)$  be the equation (45) divided by  $T_{k-1}T_kT_{k+1}$ . By making the operations

$$[\bar{D}_{k-1}\bar{E}_k(n+d+1) - \bar{D}_k\bar{E}_{k-1}(n)] + [\bar{D}_{k+1}\bar{E}_{k-1}(n-d-1) - \bar{D}_k\bar{E}_k(n)]$$

we get, for  $k \geq \tilde{k} + 1$ , the equation

$$a_{n+d+1} - a_{n-2(d+1)} - \tilde{D}_k(a_n - a_{n-d-1}) = \sum_{l=1}^k \frac{W_{k,l}}{n - (d+1)l + 2} := Q_k^{(1)}(n), \quad (46)$$

where  $W_{k,l}$  is independent of  $n$  and

$$\tilde{D}_k = \frac{\bar{D}_k^2 + \bar{D}_k\bar{D}_{k-1} + \bar{D}_k\bar{D}_{k+1}}{\bar{D}_{k-1}\bar{D}_{k+1}}.$$

Similarly, by the operations

$$[\bar{D}_k\bar{E}(k, n+d+1) - \bar{D}_{k+1}\bar{E}(k-1, n+d+1)] + [\bar{D}_k\bar{E}(k-1, n) - \bar{D}_{k-1}\bar{E}(k, n)] \quad (47)$$

and the shift  $n \rightarrow n + (d+1)k - 1$  in (47) we obtain

$$\tilde{a}_{n+d+1} - \tilde{a}_{n-2(d+1)} - \tilde{D}_k(\tilde{a}_n - \tilde{a}_{n-d-1}) = \sum_{l=1}^k \frac{\widetilde{W}_{k,l}}{n + (d+1)l - d} := \tilde{Q}_k^{(1)}(n), \quad (48)$$

where  $\widetilde{W}_{k,l}$  is independent of  $n$ . Now, for  $k \neq \kappa \geq \tilde{k} + 1$ , the equations (46) and (48) give, respectively,

$$(\tilde{D}_\kappa - \tilde{D}_k)(a_n - a_{n-d-1}) = Q_k^{(1)}(n) - Q_\kappa^{(1)}(n) \quad (49)$$

and

$$(\tilde{D}_\kappa - \tilde{D}_k) \left( \frac{n}{n+1}a_n - \frac{n-d-1}{n-d}a_{n-d-1} \right) = \tilde{Q}_k^{(1)}(n) - \tilde{Q}_\kappa^{(1)}(n). \quad (50)$$

If  $\tilde{D}_k \neq \tilde{D}_\kappa$  for some  $k \neq \kappa \geq \tilde{k} + 1$ , then by (49) and (50) we can eliminate  $a_{n-d-1}$  to get that  $a_n$  is a rational function of  $n$ . So, by Corollary 4, we have  $T_2 = 0$ .

If  $\tilde{D}_k = D$  for  $k \geq \tilde{k} + 1$ , then (46) and (48) become, respectively,

$$a_{n+d+1} - a_{n-2(d+1)} - D(a_n - a_{n-d-1}) = Q_k^{(1)}(n) \quad (51)$$

and

$$\frac{n+d+1}{n+d+2}a_{n+d+1} - \frac{n-2d-2}{n-2d-1}a_{n-2(d+1)} - D \left( \frac{n}{n+1}a_n - \frac{n-d-1}{n-d}a_{n-d-1} \right) = \tilde{Q}_k^{(1)}(n). \quad (52)$$

The combinations  $((n+d+2)Eq(52) - (n+d+1)Eq(51))/(d+1)$  and  $((n-2d-1)Eq(52) - (n-2d-2)Eq(51))/(d+1)$  give, respectively,

$$\frac{3a_{n-2d-2}}{n-2d-1} - D \left( -\frac{a_n}{n+1} + \frac{2a_{n-d-1}}{n-d} \right) = Q_k^{(3)}(n) \quad (53)$$

and

$$\frac{3a_{n+d+1}}{n+d+2} - D \left( \frac{2a_n}{n+1} - \frac{a_{n-d-1}}{n-d} \right) = Q_k^{(4)}(n). \quad (54)$$

By shifting  $n \rightarrow n + d + 1$  in (53) we obtain

$$\frac{3a_{n-d-1}}{n-d} - D \left( -\frac{a_{n+d+1}}{n+d+2} + \frac{2a_n}{n+1} \right) = Q_k^{(3)}(n+d+1). \quad (55)$$

The coefficients  $a_{n+d+1}$  and  $a_{n-d-1}$  can be eliminated by the operations  $D \times Eq(54) - 3 Eq(55)$  and  $3 Eq(54) - D \times Eq(55)$  leaving us with

$$\frac{6D - 2D^2}{n+1} a_n + \frac{D^2 - 9}{n-d} a_{n-d-1} = Q_k^{(5)}(n) \quad (56)$$

and

$$\frac{9 - D^2}{n+d+2} a_{n+d+1} - \frac{6D - 2D^2}{n+1} a_n = Q_k^{(6)}(n). \quad (57)$$

Finally, the shifting  $n \rightarrow n - d - 1$  in (57) leads to

$$\frac{9 - D^2}{n+1} a_n - \frac{6D - 2D^2}{n-d} a_{n-d-1} = Q_k^{(6)}(n-d-1) \quad (58)$$

and the operation  $(6D - 2D^2)Eq(56) + (D^2 - 9)Eq(58)$  gives

$$[(6D - 2D^2)^2 + (D^2 - 9)^2] a_n = Q_k^{(7)}(n). \quad (59)$$

According to manipulations made above,  $Q_k^{(7)}(n)$  is a rational function of  $n$ . As consequence, if  $D \neq 3$ ,  $a_n$  is a rational function of  $n$  and then  $T_2 = 0$ .

Now, we explore the case  $D = 3$ . According to the left-hand sides of (54) and (55), we have

$$Q_k^{(3)}(n+d+1) = Q_k^{(4)}(n),$$

which can be written as

$$(n+2d+2)Q_k^{(1)}(n+d+1) - (n-2d-2)Q_k^{(1)}(n) = (n+2d+3)\tilde{Q}_k^{(1)}(n+d+1) - (n-2d-1)\tilde{Q}_k^{(1)}(n). \quad (60)$$

By using, from (46) and (48), the expressions of  $Q_k^{(1)}(n)$  and  $\tilde{Q}_k^{(1)}(n)$  with  $W_{k,k+1} = \widetilde{W}_{k,k+1} = W_{k,0} = \widetilde{W}_{k,0} = 0$  we obtain

$$\sum_{l=0}^k \frac{((d+1)l+2d)W_{k,l+1} - ((d+1)l-2d-4)W_{k,l}}{n - (d+1)l + 2} = \sum_{l=0}^k \frac{(d+1)((2-l)\widetilde{W}_{k,l} - (l+2)\widetilde{W}_{k,l+1})}{n + (d+1)l + 1}. \quad (61)$$

Observe that in (61) the singularities of the left hand side are different from those of the right hand side. So,

$$((d+1)l+2d)W_{k,l+1} - ((d+1)l-2d-4)W_{k,l} = (2-l)\widetilde{W}_{k,l} - (l+2)\widetilde{W}_{k,l+1} = 0, \quad (0 \leq l \leq k), \quad (62)$$

and by induction on  $l$ , all the  $W_{k,l}$  and  $\widetilde{W}_{k,l}$  are null. Thus, (46) reads

$$a_{n+d+1} - a_{n-2(d+1)} - 3(a_n - a_{n-d-1}) = 0. \quad (63)$$

For  $n = (d+1)m + r$ ,  $m \geq 0$  and  $0 \leq r \leq d$ , the solutions of (63) have the form

$$a_{(d+1)m+r} = C_{0,r} + C_{1,r}m + C_{2,r}m^2, \quad (64)$$

where  $C_{0,r}$ ,  $C_{1,r}$  and  $C_{2,r}$  are constants. So, by Corollary 4 we get  $T_2 = 0$ .

**Case 2:** There exists  $k_0 \geq 3$  such that  $D_k = 0$  for  $k \geq k_0$ .

Suppose that  $D_k = T_k^2 - T_{k-1}T_{k+1} = 0$  for all  $k \geq k_0$ . First, notice that if there exists a  $k_1 \geq k_0$  such that  $T_{k_1} = 0$ , then  $T_{k_1-1}T_{k_1+1} = 0$ . Then,  $T_{k_1-1} = 0$  or  $T_{k_1+1} = 0$  and by Corollary 3,  $T_2 = 0$ . We have also  $T_{k_0-1} \neq 0$ , otherwise  $T_{k_0} = 0$  and by Corollary 3,  $T_2 = 0$ .

Now, for  $T_k \neq 0$  ( $k \geq k_0 - 1$ ), we have

$$\frac{T_{k+1}}{T_k} = \frac{T_k}{T_{k-1}} = \frac{T_{k_0}}{T_{k_0-1}}. \quad (65)$$

This means that

$$T_k = \left( \frac{T_{k_0}}{T_{k_0-1}} \right)^{k-k_0} T_{k_0} = ab^k \quad (66)$$

where  $a = T_{k_0}^{k_0}/T_{k_0-1}^{k_0-1} \neq 0$  and  $b = T_{k_0}/T_{k_0-1} \neq 0$ .

The substitution  $T_k = ab^k$  in (42) for  $k \geq k_0$  leads to the equation

$$\begin{aligned} & b \left( a_n - \frac{n-k(d+1)-d}{n-k(d+1)-d+1} a_{n-k(d+1)-d} \right) + \frac{n+2}{n-d+1} c_n^d - \frac{n-k(d+1)+1}{n-k(d+1)+2} c_{n-k(d+1)+1}^d \\ &= \frac{b^{-k}}{a} \sum_{l=0}^{k-1} \frac{T_{l+1}T_{k-l}}{n-l(d+1)-d+1} = Q_k(n). \end{aligned} \quad (67)$$

Let denote (67) by  $\widetilde{E}(k, n)$  and make the subtraction  $\widetilde{E}(k+1, n+d+1) - \widetilde{E}(k, n)$  to get

$$b(a_{n+d+1} - a_n) + \frac{n+d+3}{n+2} c_{n+d+1}^d - \frac{n+2}{n-d+1} c_n^d = Q_{k+1}(n+d+1) - Q_k(n). \quad (68)$$

On the right hand side of (68) we have, for  $k \geq k_0$ , the expression

$$\begin{aligned}
\tilde{Q}_k(n) &:= Q_{k+1}(n+d+1) - Q_k(n) = \frac{b^{-k-1}}{a} \sum_{l=0}^k \frac{T_{l+1}T_{k+1-l}}{n+d+1-l(d+1)-d+1} - \frac{b^{-k}}{a} \sum_{l=0}^{k-1} \frac{T_{l+1}T_{k-l}}{n-l(d+1)-d+1} \\
&= \frac{b^{-k-1}}{a} \sum_{l=0}^k \frac{T_{l+1}T_{k+1-l}}{n-l(d+1)+2} - \frac{b^{-k}}{a} \sum_{l=1}^k \frac{T_l T_{k+1-l}}{n-l(d+1)+2} \\
&= \frac{b^{-k-1}}{a} \frac{T_{k+1}T_1}{n+2} + \frac{b^{-k-1}}{a} \frac{T_k(T_2-bT_1)}{n-d+1} + \frac{b^{-k-1}}{a} \sum_{l=2}^k \frac{T_{k+1-l}(T_{l+1}-bT_l)}{n-l(d+1)+2} \\
&= \frac{T_1}{n+2} + \frac{T_2-bT_1}{b(n-d+1)} + \frac{b^{-k-1}}{a} \sum_{l=2}^k \frac{T_{k+1-l}(T_{l+1}-bT_l)}{n-l(d+1)+2}.
\end{aligned} \tag{69}$$

from which we deduce

$$\tilde{Q}_{k+1}(n) = \frac{T_1}{n+2} + \frac{T_2-bT_1}{b(n-d+1)} + \frac{b^{-k-2}}{a} \sum_{l=2}^{k+1} \frac{T_{k+2-l}(T_{l+1}-bT_l)}{n-l(d+1)+2}. \tag{70}$$

Now since the left hand side of equation (68) is independent of  $k$ , it follows

$$\tilde{Q}_{k+1}(n) - \tilde{Q}_k(n) = \frac{b^{-k-2}}{a} \sum_{l=2}^k \frac{(T_{k-l+2}-bT_{k-l+1})(T_{l+1}-bT_l)}{n-l(d+1)+2} = 0. \tag{71}$$

As a result, for  $2 \leq l \leq k$  and  $k \geq k_0$ , we have

$$(T_{k-l+2}-bT_{k-l+1})(T_{l+1}-bT_l) = 0. \tag{72}$$

Let take  $k = 2(k_0 - 2) - 1$  and  $l = k_0 - 2$  to get  $(T_{k_0-1} - bT_{k_0-2})^2 = 0$  and then  $T_{k_0-1} = bT_{k_0-2}$ , (or equivalently  $D_{k_0-1} = 0$ ). Thus, the equations (65) and (66) are valid for  $k = k_0 - 1$  and by induction we arrive at  $T_4 = bT_3$ , (or equivalently  $D_4 = 0$ ). For  $k = 4$ , the right-hand side of (45) is null. Consequently,  $V_{4,2} = 0$  and using  $T_5 = T_4^2/T_3$  (from  $D_4 = 0$ ) we get  $D_3 = 0$ . On the other side (when  $T_2 \neq 0$ ) we can write

$$T_k = \left(\frac{T_3}{T_2}\right)^{k-2} T_2 = ab^k, \text{ for } k \geq 2,$$

where  $b = T_3/T_2 \neq 0$  and  $a = T_2^3/T_3^2 \neq 0$ . Therefore, the equation (67) reads

$$\begin{aligned}
&b \left( a_n - \frac{n-k(d+1)-d}{n-k(d+1)-d+1} a_{n-k(d+1)-d} \right) + \frac{n+2}{n-d+1} c_n^d - \frac{n-k(d+1)+1}{n-k(d+1)+2} c_{n-k(d+1)+1}^d \\
&= \frac{T_1}{n-(d+1)k+2} + \frac{T_1}{n} + \sum_{l=1}^{k-2} \frac{ab}{n-l(d+1)-d+1}, \quad k \geq 2 \text{ and } n \geq (d+1)k+d.
\end{aligned} \tag{73}$$

When  $n = (d+1)k+d$  and  $n = (d+1)(k+1)$ , the equation (73) gives

$$ba_{(d+1)k+d} + \frac{(d+1)k+d+2}{(d+1)k+1} c_{(d+1)k+d}^d = \frac{d+1}{d+2} c_{d+1}^d + \frac{T_1}{d+2} + \frac{T_1}{(d+1)k+d} + \sum_{l=1}^{k-2} \frac{ab}{(d+1)(k-l)+1} \tag{74}$$



and

$$\begin{aligned}
ba_{(d+1)(k+1)} + \frac{(d+1)(k+1)+2}{(d+1)(k+1)-d+1} c_{(d+1)(k+1)}^d &= \frac{a_1 b}{d+1} + \frac{d+2}{d+3} c_{d+2}^d + \frac{T_1}{d+3} + \frac{T_1}{(d+1)(k+1)} \\
&+ \sum_{l=1}^{k-2} \frac{ab}{(d+1)(k+1-l)-d+1} \quad (75)
\end{aligned}$$

respectively. Let take  $n = (d+1)N + d$  in (73) and use (74) to obtain the expression

$$\begin{aligned}
&\frac{d+1}{d+2} c_{d+1}^d + \frac{T_1}{d+2} + \frac{T_1}{(d+1)N+d} + \sum_{l=1}^{N-2} \frac{ab}{(d+1)(N-l)+1} \\
&- \frac{(d+1)(N-k)b}{(d+1)(N-k)+1} a_{(d+1)(N-k)} - \frac{(d+1)(N-k+1)}{(d+1)(N-k)+d+2} c_{(d+1)(N-k+1)}^d \\
&= \frac{T_1}{(d+1)(N-k)+d+2} + \frac{T_1}{(d+1)N+d} + \sum_{l=1}^{k-2} \frac{ab}{(d+1)(N-l)+1}.
\end{aligned}$$

In this last equality let put  $N-k$  instead of  $k$  to get

$$\begin{aligned}
&-\frac{b(d+1)k}{(d+1)k+1} a_{(d+1)k} - \frac{(d+1)(k+1)}{(d+1)(k+1)+1} c_{(d+1)(k+1)}^d = \\
&= -\frac{(d+1)}{d+2} c_{d+1}^d - \frac{T_1}{d+2} - \sum_{l=1}^{N-2} \frac{ab}{(d+1)(N-l)+1} + \frac{T_1}{(d+1)k+d+2} + \sum_{l=1}^{N-k-2} \frac{ab}{(d+1)(N-l)+1} \\
&= -\frac{d+1}{d+2} c_{d+1}^d - \frac{T_1}{d+2} + \frac{T_1}{(d+1)k+d+2} - \sum_{l=2}^{k+1} \frac{ab}{(d+1)l+1}. \quad (76)
\end{aligned}$$

After defining  $A_1 = \frac{a_1}{d+1} + \frac{d+2}{(d+3)b} c_{d+2}^d + \frac{T_1}{(d+3)b}$ ,  $A_2 = -\frac{d+1}{d+2} \frac{c_{d+1}^d}{b} - \frac{T_1}{(d+2)b}$  and  $A_3 = \frac{T_1}{b}$ , the operation

$$\frac{1}{(d+1)(k+1)+2} \left( \frac{(d+1)(k+1)}{(d+1)(k+1)+1} Eq(75) + \frac{(d+1)(k+1)+2}{(d+1)(k+1)-d+1} Eq(76) \right)$$

leads to

$$\begin{aligned}
& \frac{(d+1)(k+1)}{((d+1)(k+1)+2)((d+1)(k+1)+1)} a_{(d+1)(k+1)} - \frac{(d+1)k}{((d+1)k+2)((d+1)k+1)} a_{(d+1)k} = \\
& = \frac{-A_1 d + d A_3 - A_3}{((d+1)k + d + 2) d} + \frac{2 A_1 - A_3}{(d+1)k + d + 3} + \frac{A_2 d + A_3}{((d+1)k + 2) d} \\
& + \left( \frac{2}{(d+1)k + d + 3} - \frac{1}{(d+1)k + d + 2} \right) \sum_{l=1}^{k-2} \frac{a}{(d+1)(k+1-l) - d + 1} \\
& - \frac{1}{(d+1)(k+1) - d + 1} \sum_{l=2}^{k+1} \frac{a}{(d+1)l + 1} \\
& = \frac{-A_1 d + d A_3 - A_3}{((d+1)k + d + 2) d} + \frac{2 A_1 - A_3}{(d+1)k + d + 3} + \frac{A_2 d + A_3}{((d+1)k + 2) d} \\
& + \left( \frac{2}{(d+1)k + d + 3} - \frac{1}{(d+1)k + d + 2} \right) \sum_{l=2}^{k-1} \frac{a}{(d+1)(l+1) - d + 1} \\
& - \frac{1}{(d+1)(k+1) - d + 1} \sum_{l=2}^{k+1} \frac{a}{(d+1)l + 1} \\
& = \frac{B_1}{k + \frac{d+3}{d+1}} + \frac{B_2}{k + \frac{d+2}{d+1}} + \frac{B_3}{k + \frac{2}{d+1}} + \frac{a}{(d+1)^2} \left( \frac{2}{k + \frac{d+3}{d+1}} - \frac{1}{k + \frac{d+2}{d+1}} \right) \Psi \left( k + \frac{2}{d+1} \right) \\
& - \frac{a}{(d+1)^2} \frac{1}{k + \frac{2}{d+1}} \Psi \left( k + 2 + \frac{1}{d+1} \right), \tag{77}
\end{aligned}$$

where the Digamma function  $\Psi(x)$  as well as the short notations  $B_1 = \frac{2A_1 - A_3}{d+1} - \frac{2a}{(d+1)^2} \Psi \left( 2 + \frac{2}{d+1} \right)$ ,  $B_2 = \frac{-dA_1 + (d-1)A_3}{d(d+1)} + \frac{a}{(d+1)^2} \Psi \left( 2 + \frac{2}{d+1} \right)$  and  $B_3 = \frac{dA_2 + A_3}{d(d+1)} + \frac{a}{(d+1)^2} \Psi \left( 2 + \frac{1}{d+1} \right)$  are introduced. Taking

$$U_k = \frac{(d+1)k}{((d+1)k+2)((d+1)k+1)} a_{(d+1)k}$$

and

$$\begin{aligned}
G(k+1) &= \frac{B_1}{k + \frac{d+3}{d+1}} + \frac{B_2}{k + \frac{d+2}{d+1}} + \frac{B_3}{k + \frac{2}{d+1}} + \frac{a}{(d+1)^2} \left( \frac{2}{k + \frac{d+3}{d+1}} - \frac{1}{k + \frac{d+2}{d+1}} \right) \Psi \left( k + \frac{2}{d+1} \right) \\
&- \frac{a}{(d+1)^2} \frac{1}{k + \frac{2}{d+1}} \Psi \left( k + 2 + \frac{1}{d+1} \right). \tag{78}
\end{aligned}$$

then (77) can be written in compact form as

$$U_{k+1} - U_k = G(k+1).$$

The later recurrence is easily solved to give

$$U_k = U_3 + \sum_{j=4}^k G(j).$$

By using the formula  $\Psi(j+1) = \Psi(j) + 1/j$  and the relations [13, Theorems 3.1 and 3.2]

$$\sum_{l=0}^k \frac{\Psi(l+\alpha)}{l+\beta} + \sum_{l=0}^k \frac{\Psi(l+\beta+1)}{l+\alpha} = \Psi(k+\alpha+1)\Psi(k+\beta+1) - \Psi(\alpha)\Psi(\beta), \quad (79)$$

$$\sum_{j=0}^k \frac{\Psi(j+\beta)}{j+\beta} = \frac{1}{2} [\Psi'(k+\beta+1) - \Psi'(\beta) + \Psi(k+\beta+1)^2 - \Psi(\beta)^2], \quad (80)$$

we obtain

$$\begin{aligned} U_k &= \frac{(d+1)k}{((d+1)k+2)((d+1)k+1)} a_{(d+1)k} = B_1 \Psi\left(k + \frac{d+3}{d+1}\right) + B_2 \Psi\left(k + \frac{d+2}{d+1}\right) + B_3 \Psi\left(k + \frac{2}{d+1}\right) \\ &+ \frac{2a}{(d+1)((d+1)k+2)} + \frac{a}{(d+1)^2} \left( \Psi'\left(k + \frac{d+3}{d+1}\right) + \left( \Psi\left(k + \frac{d+3}{d+1}\right) \right)^2 \right) \\ &- \frac{a}{(d+1)^2} \Psi\left(k + \frac{2}{d+1}\right) \Psi\left(k + \frac{d+2}{d+1}\right) + \delta_2, \end{aligned} \quad (81)$$

where

$$\begin{aligned} \delta_2 &= \frac{a}{(d+1)^2} \Psi\left(\frac{2}{d+1}\right) \Psi\left(\frac{d+2}{d+1}\right) - B_1 \Psi\left(\frac{d+3}{d+1}\right) - B_2 \Psi\left(\frac{d+2}{d+1}\right) - B_3 \Psi\left(\frac{2}{d+1}\right) - \\ &\frac{a}{d+1} - \frac{a}{(d+1)^2} \Psi'\left(\frac{d+3}{d+1}\right) - \frac{a}{(d+1)^2} \left( \Psi\left(\frac{d+3}{d+1}\right) \right)^2 - G(1) - G(2) - G(3) + U_3. \end{aligned}$$

From (81) we deduce the asymptotic behaviour of  $a_{(d+1)k}$  as  $k \rightarrow \infty$ :

$$\begin{aligned} a_{(d+1)k} &= \left( \delta_1 \left( k + \frac{3}{d+1} + \frac{2}{(d+1)^2 k} \right) + \frac{a(d+2)}{(d+1)^2} + \frac{1}{2} \frac{a(d+4)}{(d+1)^3 k} + \frac{1}{6} \frac{ad(d+2)}{(d+1)^4 k^2} - \frac{1}{4} \frac{ad(d+2)}{(d+1)^5 k^3} + \dots \right) \ln(k) \\ &+ \delta_2(d+1)k + \delta_3 + \frac{\delta_4}{k} + \frac{\delta_5}{k^2} + \frac{\delta_6}{k^3} + \dots \end{aligned} \quad (82)$$

where coefficients  $\delta_i$  are defined by (higher terms are omitted)

$$\begin{aligned} \delta_1 &= (d+1)(B_1 + B_2 + B_3), \\ \delta_3 &= \frac{d^2 + d + 4}{d+1} B_1 + \frac{d^2 + 3}{d+1} B_2 - \frac{d-3}{d+1} B_3 + 3\delta_2 + \frac{3}{d+1} a, \\ \delta_4 &= \frac{5d^2 + 4d + 47}{6(d+1)^2} B_1 + \frac{8d^2 + d + 41}{6(d+1)^2} B_2 - \frac{d^2 + 8d - 41}{6(d+1)^2} B_3 + \frac{2}{d+1} \delta_2 + \frac{9d + 13}{2(d+1)^3} a, \\ \delta_5 &= \frac{d^2 + 2d + 9}{6(d+1)^3} B_1 - \frac{d^3 + 2d^2 - 9}{6(d+1)^3} B_2 + \frac{d^2 + 2d + 9}{6(d+1)^3} B_3 + \frac{d^3 + 5d^2 - 2d + 6}{12(d+1)^4} a, \\ \delta_6 &= \frac{d^2 + 2d - 19}{60(d+1)^2} B_1 + \frac{d^3 + 18d^2 + 13d - 19}{60(d+1)^3} B_2 + \frac{d^2 + 2d - 19}{60(d+1)^2} B_3 - \frac{7d^3 + 20d^2 - 11d - 8}{24(d+1)^5} a, \\ &\vdots \end{aligned}$$

At this level we should remark that  $\lim_{k \rightarrow \infty} a_{2k} = \infty$  for all  $\delta_i$ ,  $i = 1, 2, 3, \dots$ , since  $a \neq 0$ . Recall that  $c_n^d = \frac{T_1}{d+1} \left( (n-d+1) \frac{a_n}{a_{n-d}} - (n-d) \right)$ , for  $n \geq d$ , then the equation (74) can be written as

$$a_{(d+1)k+d} \left( b + \frac{T_1}{d+1} \frac{(d+1)k+d+2}{a_{(d+1)k}} \right) = \phi(k), \quad (83)$$

where

$$\phi(k) = \frac{T_1}{d+1} \frac{((d+1)k)((d+1)k+d+2)}{(d+1)k+1} - bA_2 + \frac{bA_3}{(d+1)k+d} + \sum_{l=2}^{k-1} \frac{ab}{(d+1)l+1},$$

and the equation (75) can be written

$$a_{(d+1)(k+1)} \left( b + \frac{T_1}{d+1} \frac{(d+1)(k+1)+2}{a_{(d+1)k+1}} \right) = \tilde{\phi}(k), \quad (84)$$

where

$$\tilde{\phi}(k) = \frac{T_1}{d+1} \frac{((d+1)(k+1)+2)((d+1)k+1)}{(d+1)k+2} + bA_1 + \frac{bA_3}{(d+1)(k+1)} + \sum_{l=3}^k \frac{ab}{(d+1)l-d+1}.$$

From (83) and (84) we have

$$\frac{1}{a_{(d+1)(k+1)+d}} = \frac{1}{\phi(k+1)} \left( b + \frac{T_1}{d+1} \frac{(d+1)(k+1)+d+2}{a_{(d+1)(k+1)}} \right) \quad (85)$$

and

$$\frac{1}{a_{(d+1)k+1}} = \frac{d+1}{T_1} \frac{1}{(d+1)(k+1)+2} \left( \frac{\tilde{\phi}(k)}{a_{(d+1)(k+1)}} - b \right), \quad (86)$$

which give an explicit formula for  $a_{(d+1)(k+1)+d}$  and  $a_{(d+1)k+1}$ .

- If we suppose  $\lim_{k \rightarrow \infty} \frac{a_{(d+1)k}}{(d+1)k} = \infty$ , then from (85) and (86) we deduce on one side

$$\lim_{k \rightarrow \infty} \frac{(d+1)(k+2)}{a_{(d+1)(k+1)+d}} = \lim_{k \rightarrow \infty} \frac{(d+1)(k+2)}{\phi(k+1)} \left( b + \frac{T_1}{d+1} \frac{(d+1)(k+1)+d+2}{a_{(d+1)(k+1)}} \right) = \frac{(d+1)b}{T_1} \quad (87)$$

and

$$\lim_{k \rightarrow \infty} \frac{(d+1)(k+1)+2}{a_{(d+1)k+1}} = \lim_{k \rightarrow \infty} \frac{d+1}{T_1} \left( \frac{\tilde{\phi}(k)}{a_{(d+1)(k+1)}} - b \right) = -\frac{(d+1)b}{T_1}. \quad (88)$$

On the other side, for  $n = (d+1)k + 2d + 1$ , (41) reads

$$\frac{(d+1)T_2}{T_1^2} \left( 1 - \frac{(d+1)k}{(d+1)k+1} \frac{a_{(d+1)k}}{a_{(d+1)(k+1)+d}} \right) = \frac{(d+1)(k+2)}{a_{(d+1)(k+1)+d}} - \frac{2((d+1)(k+1)+1)}{a_{(d+1)(k+1)}} + \frac{(d+1)k+2}{a_{(d+1)k+1}}. \quad (89)$$

Under the assumption  $T_2 \neq 0$ , (89) admits the limit  $\infty = 0$ , as  $k \rightarrow \infty$ , which exhibit a contradiction.

- Now if  $\lim_{k \rightarrow \infty} \frac{a_{(d+1)k}}{(d+1)k} = \eta_1 \neq 0$ , then from (85) and (86) we have

$$\lim_{k \rightarrow \infty} \frac{(d+1)(k+2)}{a_{(d+1)(k+1)+d}} = \lim_{k \rightarrow \infty} \frac{(d+1)(k+2)}{\phi(k+1)} \left( b + \frac{T_1}{d+1} \frac{(d+1)(k+1)+d+2}{a_{(d+1)(k+1)}} \right) = \frac{(d+1)b}{T_1} + \frac{1}{\eta_1} := \eta_2, \quad (90)$$

and

$$\lim_{k \rightarrow \infty} \frac{(d+1)(k+1)+2}{a_{(d+1)k+1}} = \lim_{k \rightarrow \infty} \frac{d+1}{T_1} \left( \frac{\tilde{\phi}(k)}{a_{(d+1)(k+1)}} - b \right) = \frac{1}{\eta_1} - \frac{(d+1)b}{T_1} = \frac{2}{\eta_1} - \eta_2. \quad (91)$$

By taking the limit in (89) we obtain,

$$\frac{(d+1)T_2}{T_1^2} (1 - \eta_1 \eta_2) = \eta_2 - \frac{2}{\eta_1} + \left( \frac{2}{\eta_1} - \eta_2 \right) = 0.$$

If  $\eta_2 = 0$ , we have the contradiction  $T_2 = 0$ . But if  $\eta_2 \neq 0$ , then  $\eta_2 = 1/\eta_1$ . From (90) we get  $(d+1)b/T_1 + 1/\eta_1 = \eta_2$ , which gives the contradiction  $b = 0$ .

- Finally if  $\lim_{k \rightarrow \infty} \frac{a_{(d+1)k}}{(d+1)k} = 0$ , then according to (82), we have  $\delta_1 = \delta_2 = 0$  and

$$\begin{aligned} a_{(d+1)k} &= \frac{((d+1)k+2)((d+1)k+1)}{(d+1)k} \left( B_1 \Psi \left( k + \frac{d+3}{d+1} \right) + B_2 \Psi \left( k + \frac{d+2}{d+1} \right) - (B_1 + B_2) \Psi \left( k + \frac{2}{d+1} \right) \right. \\ &\quad \left. + \frac{2a}{(d+1)((d+1)k+2)} + \frac{a}{(d+1)^2} \left( \Psi' \left( k + \frac{d+3}{d+1} \right) + \left( \Psi \left( k + \frac{d+3}{d+1} \right) \right)^2 \right) \right. \\ &\quad \left. - \frac{a}{(d+1)^2} \Psi \left( k + \frac{2}{d+1} \right) \Psi \left( k + \frac{d+2}{d+1} \right) \right), \\ &= \left( \frac{a(d+2)}{(d+1)^2} + \frac{1}{2} \frac{a(d+4)}{(d+1)^3 k} + \frac{1}{6} \frac{ad(d+2)}{(d+1)^4 k^2} - \frac{1}{4} \frac{ad(d+2)}{(d+1)^5 k^3} + \dots \right) \ln(k) + \delta_3 + \frac{\delta_4}{k} + \frac{\delta_5}{k^2} + \frac{\delta_6}{k^3} + \dots. \end{aligned} \quad (92)$$

Let write the equation (89) as

$$\frac{(d+1)T_2}{T_1^2} \left( 1 - \frac{(d+1)k}{(d+1)k+1} \frac{a_{(d+1)k}}{a_{(d+1)(k+1)+d}} \right) - \frac{(d+1)(k+2)}{a_{(d+1)(k+1)+d}} + \frac{2((d+1)(k+1)+1)}{a_{(d+1)(k+1)}} - \frac{(d+1)k+2}{a_{(d+1)k+1}} = 0. \quad (93)$$

After multiplying the both sides of the equation (93) by  $\phi(k+1)a_{(d+1)(k+1)}$  and using (92), (85) and (86), we get, as  $k \rightarrow \infty$ ,

$$\lambda_1 \ln(k)^2 + \lambda_2 \ln(k) + \lambda_3 + \dots = 0 \quad (94)$$

where

$$\lambda_1 = \left( -\frac{(d+2)b}{(d+1)^3 T_1^2} T_2 + \frac{b^2}{(d+1)^2 T_1} \right) a^2$$

and

$$\begin{aligned} \lambda_2 &= \left( \frac{2b(d+2)a^2}{(d+1)^3 T_1^2} \Psi \left( \frac{2d+3}{d+1} \right) - 3 \frac{(d+3)ba^2}{(d+1)^2 T_1^2} + 2 \frac{bA_2(d+2) + T_1(d+3)}{(d+1)^2 T_1^2} a \right) T_2 \\ &\quad - \frac{2b^2 a^2}{(d+1)^2 T_1} \Psi \left( \frac{2d+3}{d+1} \right) + 3 \frac{b^2 a^2}{(d+1) T_1} - 2 \frac{(bA_2 + T_1)ba}{(d+1) T_1}. \end{aligned}$$

So, we must have  $\lambda_i = 0$ ,  $i = 1, 2, 3, \dots$ . As  $a \neq 0$  and  $b \neq 0$ , from  $\lambda_1 = 0$  we get

$$T_2 = \frac{d+1}{d+2} b T_1, \quad (95)$$

and by replacing  $a = T_2/b^2$  and (95) in the equation  $\lambda_2 = 0$  we obtain

$$\frac{d-1}{(d+2)^3} T_1^2 = 0. \quad (96)$$

If  $d \neq 1$ , (96) gives  $T_1 = 0$  which is a contradiction. The case  $d = 1$  is already treated, [14, Theorem 1].

**Case 3:** For every  $k_0 \geq 3$ , there exists infinitely many  $k, \kappa \geq k_0$  such that:  $D_k = 0$  and  $D_\kappa \neq 0$ . We take  $k_1$  and  $k_2$ ,  $k_1 \neq k_2$ , with  $D_{k_1} = 0, D_{k_1+1} \neq 0, D_{k_2} = 0$  and  $D_{k_2+1} \neq 0$  to get from (45) that

$$a_n - \frac{n - k_1(d+1) - 2d - 1}{n - k_1(d+1) - 2d} a_{n-k_1(d+1)-2d-1} \quad \text{and} \quad a_n - \frac{n - k_2(d+1) - 2d - 1}{n - k_2(d+1) - 2d} a_{n-k_2(d+1)-2d-1} \quad (97)$$

are two rational functions of  $n$ . Consequently, an analogous reasoning to that of Lemma 4 completes the proof.

In the next subsection we give some expressions concerning the sequence  $\{\gamma_n^d\}_{n \geq d}$  and the power series  $F(t)$ . The later is expressed by hypergeometric series (25).

### 3.1 Expressions for $\gamma_n^d$ and $F(t)$

**Proposition 3** *The  $d$ -symmetric  $d$ -PS,  $\{P_n\}$ , generated by (1) satisfies*

$$\begin{cases} x P_n(x) = P_{n+1}(x) + \gamma_n^d P_{n-d}(x), & n \geq 0, \\ P_{-n}(x) = 0, & 1 \leq n \leq d, \text{ and } P_0(x) = 1 \end{cases} \quad (98)$$

with

$$\gamma_n^d = \frac{T_1}{d+1} \left( (n-d+1) \frac{\alpha_{n-d+1}}{\alpha_{n+1}} - (n-d) \frac{\alpha_{n-d}}{\alpha_n} \right), \quad \text{for } n \geq d, \quad (99)$$

and for  $n = dm + r$ ,  $m \geq 1$ ,  $1 \leq r \leq d$  we have

$$\gamma_{dm+r}^d = \frac{T_1}{d+1} \frac{(dm+r)! (\beta_r(m-r-1) + (d+1)b_r)}{(d(m-1)+r)! \prod_{l=1}^r (\beta_l m + b_l) \prod_{l=r}^d (\beta_l(m-1) + b_l)} \quad (100)$$

with  $\gamma_d^d = T_1 d! / \prod_{l=1}^d b_l$ ,  $b_l = (l+1)\alpha_{l+1}/\alpha_l$  and  $\beta_l = b_{d+l} - b_l$ ,  $1 \leq l \leq d$ .

**Proof:** The equation (39) is

$$c_n^d = \frac{\alpha_n}{\alpha_{n-d}} \gamma_n^d = \frac{T_1}{d+1} \left( (n-d+1) \frac{a_n}{a_{n-d}} - (n-d) \right), \quad \text{for } n \geq d. \quad (101)$$

As  $a_n = \alpha_n/\alpha_{n+1}$  we get (99).

According to Theorem 5 we have  $R(t) = T_1 t^{d+1}/(d+1)$ . So, by Corollary 5 we obtain

$$c_n^d = \frac{\alpha_n}{\alpha_{n-d}} \gamma_n^d = \frac{T_1}{d+1} \left( (n+1) \frac{b_{n-d}}{b_n} - (n-d) \right), \quad \text{for } n \geq d+1, \quad (102)$$

with  $b_n = (n+1)/a_n = (n+1)\alpha_{n+1}/\alpha_n$  and

$$b_{md+r} = \beta_r m + b_r, \quad \text{for } m \geq 0, 1 \leq r \leq d, \quad (103)$$

where  $\beta_r = b_{d+r} - b_r$ , for  $1 \leq r \leq d$ .

The equation (102) gives

$$\gamma_n^d = \frac{T_1}{d+1} \frac{\alpha_{n-d}((n+1)b_{n-d} - (n-d)b_n)}{\alpha_n b_n}, \quad \text{for } n \geq d+1. \quad (104)$$

We calculate  $\alpha_n b_n / \alpha_{n-d}$  by using the relation  $\alpha_n = b_{n-1} \alpha_{n-1} / n = \prod_{l=0}^{n-1} b_l / n!$  to find

$$\frac{\alpha_n b_n}{\alpha_{n-d}} = \frac{(n-d)!}{n!} \frac{\prod_{l=0}^n b_l}{\prod_{l=0}^{n-d-1} b_l} = \frac{(n-d)!}{n!} \prod_{l=n-d}^n b_l. \quad (105)$$

Now for  $n = md + r$ , we can write

$$\prod_{l=n-d}^n b_l = \prod_{l=r}^d b_{(m-1)d+l} \prod_{l=1}^r b_{md+l} = \prod_{l=r}^d (\beta_l(m-1) + b_l) \prod_{l=1}^r (\beta_l m + b_l) \quad (106)$$

and

$$\begin{aligned} (n+1)b_{n-d} - (n-d)b_n &= (md+r+1)b_{(m-1)d+r} - (md+r-d)b_{md+r} \\ &= (md+r+1)(\beta_r(m-1) + b_r) - (md+r-d)(\beta_r m + b_r) \\ &= \beta_r(m-r-1) + (d+1)b_r. \end{aligned} \quad (107)$$

Finally, (99) gives

$$\gamma_d^d = \frac{T_1 \alpha_1}{(d+1)\alpha_{d+1}} = \frac{T_1}{d+1} \prod_{l=1}^d \frac{\alpha_l}{\alpha_{l+1}} = \frac{T_1 d!}{\prod_{l=1}^d b_l}$$

and (100) follows by combining (105), (106) and (107). ■

**Proposition 4** *If  $\prod_{l=1}^d \beta_l \neq 0$  then  $F(t) = 1 + F_1(t) + F_2(t)$ , where*

$$F_1(t) = \alpha_d t^d {}_dF_d \left( \begin{matrix} 1, & \frac{b_d}{\beta_d}, & \left( \frac{b_{l+d}}{\beta_l} \right)_{l=1}^{d-1} \\ \left( \frac{l+d}{d} \right)_{l=1}^d \end{matrix} ; \left( \prod_{l=1}^d \beta_l \right) \left( \frac{t}{d} \right)^d \right) \quad (108)$$

and

$$F_2(t) = \sum_{r=1}^{d-1} \alpha_r t^r {}_dF_{d-1} \left( \begin{matrix} \left( \frac{b_{l+d}}{\beta_l} \right)_{l=1}^{r-1}, & \left( \frac{b_l}{\beta_l} \right)_{l=r}^d \\ \left( \frac{l}{d} \right)_{l=r+1}^{d-1}, & \left( \frac{l}{d} \right)_{l=d+1}^{d+r} \end{matrix} ; \left( \prod_{l=1}^d \beta_l \right) \left( \frac{t}{d} \right)^d \right). \quad (109)$$

Furthermore, if  $\tilde{\beta}_d := b_d - \beta_d = 2b_d - b_{2d} \neq 0$ , then  $F_1(t)$  in (108) can be written as

$$F_1(t) = \frac{\alpha_1}{\tilde{\beta}_d} \left[ {}_dF_{d-1} \left( \begin{matrix} \frac{\tilde{\beta}_d}{\beta_d}, & \left( \frac{b_l}{\beta_l} \right)_{l=1}^{d-1} \\ \left( \frac{l}{d} \right)_{l=1}^{d-1} \end{matrix} ; \left( \prod_{l=1}^d \beta_l \right) \left( \frac{t}{d} \right)^d \right) - 1 \right]. \quad (110)$$

**Proof:**

Recall that  $\alpha_n = \prod_{l=0}^{n-1} b_l/n!$ . If  $\prod_{l=1}^d \beta_l \neq 0$ , then for  $n = md + r$  and using the expression of  $b_{md+r}$ , we obtain

$$\begin{aligned}
\prod_{l=0}^{md+r-1} b_l &= b_0 b_1 b_2 \cdots b_{r-1} b_r b_{r+1} \cdots b_{d+r-1} \cdots b_{(m-1)d+r-1} b_{(m-1)d+r} \cdots b_{md} b_{md+1} \cdots b_{md+r-1} \\
&= (b_0 b_1 b_2 \cdots b_{r-1}) (b_r b_{r+d} \cdots b_{r+(m-1)d}) (b_{r+1} \cdots b_{r+1+(m-1)d}) \cdots (b_{d+r-1} b_{2d+r-1} \cdots b_{md+r-1}) \\
&= \prod_{l=0}^{r-1} b_l \prod_{l=r}^{d+r-1} b_l b_{l+d} \cdots b_{l+(m-1)d} \\
&= r! \alpha_r \prod_{l=r}^d b_l b_{l+d} \cdots b_{l+(m-1)d} \prod_{l=d+1}^{d+r-1} b_l b_{l+d} \cdots b_{l+(m-1)d} \\
&= r! \alpha_r \left( \prod_{l=r}^d \beta_l \right)_m^m \prod_{l=r}^d \left( \frac{b_l}{\beta_l} \right)_m \left( \prod_{l=1}^{r-1} \beta_l \right)_m^{r-1} \prod_{l=1}^{r-1} \left( 1 + \frac{b_l}{\beta_l} \right)_m \\
&= r! \alpha_r \left( \prod_{l=1}^d \beta_l \right)_m^m \prod_{l=r}^d \left( \frac{b_l}{\beta_l} \right)_m \prod_{l=1}^{r-1} \left( \frac{b_{l+d}}{\beta_l} \right)_m. \tag{111}
\end{aligned}$$

We have also, for  $m \geq 0$  and  $0 \leq r \leq d-1$ , the expressions [6, Lemma 3.3]

$$(md+r)! = r! m! d^{dm} \prod_{l=r+1}^{d-1} \left( \frac{l}{d} \right)_m \prod_{l=d+1}^{d+r} \left( \frac{l}{d} \right)_m$$

and

$$((m+1)d)! = d! d^{dm} \prod_{l=1}^d \left( 1 + \frac{l}{d} \right)_m.$$

So,

$$\alpha_{md+r} = \frac{\alpha_r \prod_{l=1}^{r-1} \left( \frac{b_{l+d}}{\beta_l} \right)_m \prod_{l=r}^d \left( \frac{b_l}{\beta_l} \right)_m}{m! \prod_{l=r+1}^{d-1} \left( \frac{l}{d} \right)_m \prod_{l=d+1}^{d+r} \left( \frac{l}{d} \right)_m} \left( \left( \prod_{l=1}^d \beta_l \right) \left( \frac{1}{d} \right)^d \right)_m^m, \quad \text{for } 1 \leq r \leq d-1, \tag{112}$$

$$\alpha_{md+d} = \frac{\alpha_d \prod_{l=1}^{d-1} \left( \frac{b_{l+d}}{\beta_l} \right)_m \left( \frac{b_d}{\beta_d} \right)_m}{\prod_{l=1}^d \left( 1 + \frac{l}{d} \right)_m} \left( \left( \prod_{l=1}^d \beta_l \right) \left( \frac{1}{d} \right)^d \right)_m^m \tag{113}$$

and, if  $\tilde{\beta}_d \neq 0$ ,

$$\alpha_{md+d} = \frac{\alpha_1 \prod_{l=1}^{d-1} \left( \frac{b_l}{\beta_l} \right)_{m+1} \left( \frac{\tilde{\beta}_d}{\beta_d} \right)_{m+1}}{\tilde{\beta}_d (m+1)! \prod_{l=1}^{d-1} \left( \frac{l}{d} \right)_{m+1}} \left( \left( \prod_{l=1}^d \beta_l \right) \left( \frac{1}{d} \right)^d \right)_{m+1}^{m+1}. \tag{114}$$

Now, expanding  $F(t)$  as

$$F(t) = \sum_{n \geq 0} \alpha_n t^n = 1 + \sum_{m \geq 0} \alpha_{md+d} t^{md+d} + \sum_{r=1}^{d-1} \sum_{m \geq 0} \alpha_{md+r} t^{md+r},$$



the expressions (108), (109) and (110) follow from (112), (113) and (114), respectively.  $\blacksquare$

**Remark 4** In proposition 4 two expressions of  $F(t)$  are given. The first, when  $\prod_{l=1}^d \beta_l \neq 0$ , Equations (108) and (109), from which we can deduce the other limiting cases by tending to zero at least a constant  $\beta_r$ ,  $1 \leq r \leq d$ . So, we can enumerate  $2^d$  expressions of  $F(t)$  similar to that given in [6, Theorem 3.1]. The second, when  $\prod_{l=1}^d \beta_l \neq 0$  and  $\tilde{\beta}_d = b_d - \beta_d \neq 0$ , Equations (110) and (109), seems to be a new representation of  $F(t)$ . Similarly, the other limiting cases can be obtained by tending to zero at least a constant  $\beta_r$ ,  $1 \leq r \leq d$ , and  $\tilde{\beta}_d$ . So, in this second representation, we can enumerate  $3 \cdot 2^{d-1} (= 2^d + 2^{d-1})$  expressions of  $F(t)$ . We note that, the resulting  $2^{d-1}$  expressions when  $\tilde{\beta}_d \rightarrow 0$ , i.e.  $\beta_d \rightarrow b_d$ , are special cases of the first representation when  $\beta_d = b_d$ . See the illustrative examples given below.

**Example 1** If  $d = 1$  then  $R(t) = T_1 t^2 / 2$ ,  $b_n = (b_2 - b_1)n + 2b_1 - b_2 = \beta_1 n + \tilde{\beta}_1$ , for  $n \geq 1$ , and  $F(t)$ ,  $\gamma_n^d$  have the expressions:

1) If  $\beta_1 \neq 0$  we have

$$F(t) \equiv F^{\beta_1}(t) = 1 + \alpha_1 t {}_2F_1 \left( \begin{matrix} 1, & \frac{b_1}{\beta_1} \\ 2 \end{matrix} ; \beta_1 t \right) \quad (115)$$

with  $\gamma_1^1 = T_1 / b_1$  and for  $n \geq 2$ ,

$$\gamma_n^1 = \frac{T_1}{2} \frac{n(\beta_1(n-1) + 2\tilde{\beta}_1)}{(\beta_1 n + \tilde{\beta}_1)(\beta_1(n-1) + \tilde{\beta}_1)}.$$

The limiting case is

$$\lim_{\beta_1 \rightarrow 0} F^{\beta_1}(t) = 1 + \alpha_1 t {}_1F_1 \left( \begin{matrix} 1 \\ 2 \end{matrix} ; b_1 t \right) = 1 + \frac{\alpha_1}{b_1} (e^{b_1 t} - 1) \quad (116)$$

with  $\gamma_n^1 = T_1 n / b_1$  for  $n \geq 1$ .

2) If  $\tilde{\beta}_1 \beta_1 \neq 0$ , [2, 14],

$$F(t) \equiv F^{\beta_1, \tilde{\beta}_1}(t) = 1 + \frac{\alpha_1}{\tilde{\beta}_1} \left( (1 - \beta_1 t)^{-\frac{\tilde{\beta}_1}{\beta_1}} - 1 \right) \quad (\text{Ultraspherical polynomials}) \quad (117)$$

with

$$\gamma_n^1 = \frac{T_1}{2} \frac{n(\beta_1(n-1) + 2\tilde{\beta}_1)}{(\beta_1 n + \tilde{\beta}_1)(\beta_1(n-1) + \tilde{\beta}_1)}, \text{ for } n \geq 1.$$

The limiting cases are

$$\lim_{\beta_1 \rightarrow 0} F^{\beta_1, \tilde{\beta}_1}(t) = 1 + \frac{\alpha_1}{b_1} (e^{b_1 t} - 1) \quad (\text{Hermite polynomials}) \quad (118)$$

and

$$\lim_{\tilde{\beta}_1 \rightarrow 0} F^{\beta_1, \tilde{\beta}_1}(t) = 1 - \frac{\alpha_1}{\beta_1} \ln(1 - \beta_1 t) \quad (\text{Chebyshev polynomials of the first kind}). \quad (119)$$

Remark that (117) and (119) are special cases of (115) for  $\tilde{\beta}_1 \beta_1 \neq 0$  and  $\tilde{\beta}_1 = 0$ , i.e.  $\beta_1 = b_1$ , respectively. Also, (116) is exactly (118), since in this case  $\beta_1 \rightarrow b_1$ .

**Example 2** For  $d \geq 1$  we take  $b_n = \alpha n + \beta$  for all  $n \geq 1$ . So, from (103) we have  $\beta_r = d\alpha$  and, of course,  $b_r = \alpha r + \beta$ , for  $1 \leq r \leq d$ . Thus, for  $\alpha\beta \neq 0$ ,  $\gamma_n^d$  becomes

$$\gamma_n^d = \frac{T_1 \alpha^{-d-1}}{d+1} \frac{n! (\alpha(n-d) + (d+1)\beta)}{(n-d)! (n + \frac{\beta}{\alpha} - d)_{d+1}}, \quad n \geq d, \quad (120)$$

and for  $F(t)$ , since  $\alpha_n = \prod_{l=0}^{n-1} b_l/n!$ , we obtain (see [2] for calculations)

$$F(t) \equiv F^{\alpha, \beta}(t) = 1 + \frac{\alpha_1}{\beta} \left( (1 - \alpha t)^{-\beta/\alpha} - 1 \right). \quad (121)$$

Let  $\lambda = \beta/\alpha$ . Then for  $T_1 = \alpha^{d+1}(d+1)^{-d}$  and with the change of variable  $t \rightarrow (d+1)t/\alpha$  in the generating function  $(1 - \alpha(xt - T_1 t^{d+1}/(d+1)))^{-\lambda}$ , we meet the Humbert polynomials [11] generated by  $(1 - (d+1)xt + t^{d+1})^{-\lambda}$ . For  $d = 1$  we have the ultraspherical polynomials.

The limiting cases are

1.  $\alpha \rightarrow 0$  and  $\beta \neq 0$ :

$$\lim_{\alpha \rightarrow 0} F^{\alpha, \beta}(t) = 1 + \frac{\alpha_1}{\beta} (e^{\beta t} - 1) \quad (122)$$

with  $\gamma_n^d = T_1 \beta^{-d} n!/(n-d)!$ , for  $n \geq d$ . In the generating function  $\exp(\beta(xt - T_1 t^{d+1}/(d+1)))$ , with  $T_1 = \beta^d((d+1)d!)^{-1}$  and the change of variable  $t \rightarrow t/\beta$ , we find the generating function  $\exp(xt - (d+1)^{-2} t^{d+1}/d!)$  of the Gould-Hopper polynomials [10].

2.  $\beta \rightarrow 0$  and  $\alpha \neq 0$

$$\lim_{\beta \rightarrow 0} F^{\alpha, \beta}(t) = 1 - \frac{\alpha_1}{\alpha} \ln(1 - \alpha t) \quad (123)$$

with  $\gamma_d^d = T_1 \alpha^{-d}$  and  $\gamma_n^d = T_1 \alpha^{-d}/(d+1)$  for  $n \geq d+1$ .

Let  $b = T_1 \alpha^{-d}/(d+1)$ . Then by the shift  $n \rightarrow n+d$  in (98), these polynomials satisfy

$$\begin{cases} P_{n+d+1}(x) = xP_{n+d}(x) - bP_n(x), & n \geq 1, \\ P_n(x) = x^n, & 0 \leq n \leq d, \text{ and } P_{d+1}(x) = xP_d(x) - (d+1)bP_0(x), \end{cases} \quad (124)$$

where we recognise the monic Chebyshev  $d$ -OPS of the first kind generated by (see [5, Theorem 5.1])

$$\frac{1 - dbt^{d+1}}{1 - xt + bt^{d+1}} = \sum_{n \geq 0} P_n(x) t^n. \quad (125)$$

Remark that

$$\int_0^t \frac{1}{t} \left( \frac{1 - dbt^{d+1}}{1 - xt + bt^{d+1}} - 1 \right) dt = -\ln(1 - xt + bt^{d+1}). \quad (126)$$

Then, by changing the variable  $t \rightarrow \alpha t$ , multiplying by  $\alpha_1/\alpha$  and adding 1 in (126), we get the generating function (with  $F(t)$  as in (123)),

$$1 - \frac{\alpha_1}{\alpha} \ln \left( 1 - \alpha \left( xt - \frac{T_1}{d+1} t^{d+1} \right) \right) = 1 + \sum_{n \geq 1} \frac{\alpha_1}{\alpha n} P_n(x) t^n. \quad (127)$$

**Example 3** For  $d = 2$  we have  $R(t) = T_1 t^3 / 3$  and from (100) we get, for  $m \geq 1$ , the two expressions

$$\gamma_{2m+2}^2 = \frac{T_1}{3} \frac{2(m+1)(2m+1)(\beta_2 m + 3\tilde{\beta}_2)}{(\beta_1 m + b_1)(\beta_2 m + b_2)(\beta_2 m + \tilde{\beta}_2)} \quad (128)$$

and

$$\gamma_{2m+1}^2 = \frac{T_1}{3} \frac{2m(2m+1)(\beta_1(m-2) + 3b_1)}{(\beta_1 m + b_1)(\beta_1(m-1) + b_1)(\beta_2 m + \tilde{\beta}_2)}, \quad (129)$$

with  $\gamma_2^2 = 2T_1/(b_1 b_2)$ ,  $\beta_1 = b_3 - b_1$ ,  $\beta_2 = b_4 - b_2$  and  $\tilde{\beta}_2 = 2b_2 - b_4$ .

We enumerate the following forms of  $F(t)$ :

**A.** The first representation by (108) and (109).

If  $\beta_1 \beta_2 \neq 0$ , then

$$F(t) = 1 + \alpha_2 t^2 {}_3F_2 \left( \begin{matrix} 1, & \frac{b_2}{\beta_2}, & \frac{b_3}{\beta_1} \\ \frac{3}{2}, & 2 \end{matrix} ; \beta_1 \beta_2 \left( \frac{t}{2} \right)^2 \right) + \alpha_1 t {}_2F_1 \left( \begin{matrix} \frac{b_1}{\beta_1}, & \frac{b_2}{\beta_2} \\ \frac{3}{2} \end{matrix} ; \beta_1 \beta_2 \left( \frac{t}{2} \right)^2 \right). \quad (130)$$

The limiting cases are obtained when:  $\beta_1 \rightarrow 0$ ,  $\beta_2 \rightarrow 0$  or  $(\beta_1, \beta_2) \rightarrow (0, 0)$ .

**B.** The second representation, by (109) and (110), with its limiting cases.

1. If  $\tilde{\beta}_2 \beta_1 \beta_2 \neq 0$  we have

$$F(t) = 1 + \frac{\alpha_1}{\tilde{\beta}_2} \left[ {}_2F_1 \left( \begin{matrix} \frac{\tilde{\beta}_2}{\beta_2}, & \frac{b_1}{\beta_1} \\ \frac{1}{2} \end{matrix} ; \beta_1 \beta_2 \left( \frac{t}{2} \right)^2 \right) - 1 \right] + \alpha_1 t {}_2F_1 \left( \begin{matrix} \frac{b_1}{\beta_1}, & \frac{b_2}{\beta_2} \\ \frac{3}{2} \end{matrix} ; \beta_1 \beta_2 \left( \frac{t}{2} \right)^2 \right). \quad (131)$$

2. If  $\beta_1 \rightarrow 0$ :

$$F(t) = 1 + \frac{\alpha_1}{\tilde{\beta}_2} \left[ {}_1F_1 \left( \begin{matrix} \frac{\tilde{\beta}_2}{\beta_2} \\ \frac{1}{2} \end{matrix} ; b_1 \beta_2 \left( \frac{t}{2} \right)^2 \right) - 1 \right] + \alpha_1 t {}_1F_1 \left( \begin{matrix} \frac{b_2}{\beta_2} \\ \frac{3}{2} \end{matrix} ; b_1 \beta_2 \left( \frac{t}{2} \right)^2 \right). \quad (132)$$

with

$$\gamma_{2m}^2 = \frac{T_1}{3b_1} \frac{2m(2m-1)(\beta_2(m-1) + 3\tilde{\beta}_2)}{(\beta_2 m + \tilde{\beta}_2)(\beta_2(m-1) + \tilde{\beta}_2)}, \quad m \geq 1 \quad (133)$$

$$\gamma_{2m+1}^2 = \frac{T_1}{b_1} \frac{(2m)(2m+1)}{\beta_2 m + \tilde{\beta}_2}, \quad m \geq 1, \quad (134)$$

3. If  $\beta_2 \rightarrow 0$ :

$$F(t) = 1 + \frac{\alpha_1}{b_2} \left[ {}_1F_1 \left( \begin{matrix} \frac{b_1}{\beta_1} \\ \frac{1}{2} \end{matrix} ; b_2 \beta_1 \left( \frac{t}{2} \right)^2 \right) - 1 \right] + \alpha_1 t {}_1F_1 \left( \begin{matrix} \frac{b_1}{\beta_1} \\ \frac{3}{2} \end{matrix} ; b_2 \beta_1 \left( \frac{t}{2} \right)^2 \right). \quad (135)$$

with

$$\gamma_{2m}^2 = \frac{T_1}{b_2} \frac{(2m)(2m-1)}{\beta_1(m-1) + b_1}, \quad m \geq 1 \quad (136)$$

$$\gamma_{2m+1}^2 = \frac{T_1}{3b_2} \frac{2m(2m+1)(\beta_1(m-2) + 3b_1)}{(\beta_1 m + b_1)(\beta_1(m-1) + b_1)}, \quad m \geq 1, \quad (137)$$

4. If  $\tilde{\beta}_2 \rightarrow 0$ :

$$F(t) = 1 + \alpha_2 t^2 {}_3F_2 \left( \begin{matrix} 1 & 1 & 1 + \frac{b_1}{\beta_1} \\ 2 & \frac{3}{2} \end{matrix} ; \beta_1 b_2 \left( \frac{t}{2} \right)^2 \right) + \alpha_1 t {}_2F_1 \left( \begin{matrix} 1 & \frac{b_1}{\beta_1} \\ \frac{3}{2} \end{matrix} ; \beta_1 b_2 \left( \frac{t}{2} \right)^2 \right). \quad (138)$$

with  $\gamma_2^2 = 2T_1/(b_1 b_2)$ ,

$$\gamma_{2m}^2 = \frac{T_1}{3b_2} \frac{2(2m-1)}{(\beta_1(m-1) + b_1)}, \quad m \geq 2 \quad (139)$$

$$\gamma_{2m+1}^2 = \frac{T_1}{3b_2} \frac{2(2m+1)(\beta_1(m-2) + 3b_1)}{(\beta_1 m + b_1)(\beta_1(m-1) + b_1)}, \quad m \geq 1. \quad (140)$$

Clearly (138) is (130) with  $\beta_2 = b_2$ .

5. If  $\beta_1 \rightarrow 0$  and  $\beta_2 \rightarrow 0$ :

$$\begin{aligned} F(t) &= 1 + \frac{\alpha_1}{b_2} \left[ {}_0F_1 \left( \begin{matrix} - \\ \frac{1}{2} \end{matrix} ; b_1 b_2 \left( \frac{t}{2} \right)^2 \right) - 1 \right] + \alpha_1 t {}_0F_1 \left( \begin{matrix} - \\ \frac{3}{2} \end{matrix} ; b_1 b_2 \left( \frac{t}{2} \right)^2 \right) \\ &= 1 + \frac{\alpha_1}{b_2} \left( \cosh \left( \sqrt{b_1 b_2} t \right) - 1 + \sqrt{\frac{b_2}{b_1}} \sinh \left( \sqrt{b_1 b_2} t \right) \right). \end{aligned} \quad (141)$$

with

$$\gamma_n^2 = \frac{T_1}{b_1 b_2} n(n-1), \quad n \geq 2, \quad (142)$$

6. If  $\beta_1 \rightarrow 0$  and  $\tilde{\beta}_2 \rightarrow 0$ :

$$F(t) = 1 + \alpha_2 t^2 {}_2F_2 \left( \begin{matrix} 1 & 1 \\ 2 & \frac{3}{2} \end{matrix} ; b_1 b_2 \left( \frac{t}{2} \right)^2 \right) + \alpha_1 t {}_1F_1 \left( \begin{matrix} 1 \\ \frac{3}{2} \end{matrix} ; b_1 b_2 \left( \frac{t}{2} \right)^2 \right). \quad (143)$$

with  $\gamma_2^2 = 2T_1/(b_1 b_2)$ ,

$$\gamma_{2m}^2 = \frac{2T_1}{3b_1 b_2} (2m-1), \quad m \geq 2 \quad (144)$$

$$\gamma_{2m+1}^2 = \frac{2T_1}{b_1 b_2} (2m+1), \quad m \geq 1. \quad (145)$$

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